

\mathcal{D}^\dagger -AFFINITY OF FORMAL MODELS OF FLAG VARIETIES

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ABSTRACT. Let \mathbb{G} be a split reductive group over a finite extension L of \mathbb{Q}_p and let $G = \mathbb{G}(L)$. In this paper we prove that formal models \mathfrak{X} of the flag variety of \mathbb{G} are $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -affine for certain sheaves of arithmetic differential operators $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. Given a G -equivariant system \mathcal{F} of formal models \mathfrak{X} , we deduce that the category of admissible locally analytic G -representations with trivial central character is naturally equivalent to a full subcategory of the category of G -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules on the projective limit $\mathfrak{X}_\infty = \mathfrak{X}_\infty(\mathcal{F})$ of this system of formal models. When \mathcal{F} is cofinal in the system of all formal models, the space $\mathfrak{X}_\infty(\mathcal{F})$ can be identified with the adic flag variety of \mathbb{G} .

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1. INTRODUCTION

Let L/\mathbb{Q}_p be a finite extension with ring of integers $\mathfrak{o} = \mathfrak{o}_L$. In [18] we have introduced certain sheaves of differential operators $\tilde{\mathcal{D}}_{n,k,\mathbb{Q}}^\dagger$ on a family of semistable models \mathfrak{X}_n of the projective line and have shown that \mathfrak{X}_n is $\tilde{\mathcal{D}}_{n,k,\mathbb{Q}}^\dagger$ -affine. In this paper we generalize this approach to (not necessarily semistable) formal models of general flag varieties of split reductive groups. So let \mathbb{G}_0 be a reductive group scheme over \mathfrak{o} with generic fibre \mathbb{G} . Denote by X^{rig} the rigid analytic space attached to the flag variety of \mathbb{G} . We consider a formal model \mathfrak{X} of X^{rig} which we assume to be an admissible blow-up of a smooth formal model \mathfrak{X}_0 . On such a model \mathfrak{X} we introduce certain sheaves of differential operators $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$, for $k \geq k_{\mathfrak{X}}$, where $k_{\mathfrak{X}}$ is a non-negative integer depending on \mathfrak{X} . Our first main result is then

Theorem 1. *For all $k \geq k_{\mathfrak{X}}$ the formal scheme \mathfrak{X} is $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -affine.*

This means that the global sections functor furnishes an equivalence of categories between coherent modules over $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ and finitely presented modules over the ring $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger)$ (cf. [4],[8], [9] for the classical setting). It is shown that $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger)$ can be identified with a central reduction of Emerton's analytic distribution algebra $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)$ of the wide open rigid analytic k^{th} congruence subgroup of \mathbb{G}_0 , cf. [10].

As in [18] our main motivation for this result concerns locally analytic representations. The category of admissible locally analytic representations of the p -adic group $G := \mathbb{G}(L)^1$ with trivial infinitesimal character θ_0 is anti-equivalent to the category of coadmissible modules over $D(G)_{\theta_0}$, the central reduction of the locally L -analytic distribution algebra of G at θ_0 . On the geometric side, we consider a projective system \mathcal{F} of formal models \mathfrak{X} . We assume that each model in this system is an admissible blow-up of some smooth formal model of X^{rig} , and that the system \mathcal{F} is equipped with an action of G . Then we can form the limit of the sheaves $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ and obtain a sheaf of (infinite order) differential operators $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ on $\mathfrak{X}_\infty = \mathfrak{X}_\infty(\mathcal{F}) = \varprojlim_{\mathcal{F}} \mathfrak{X}$. In this situation, the localization functors for the various \mathfrak{X} assemble to a functor $\mathcal{L}oc_\infty^\dagger$ from the coadmissible modules over $D(G)_{\theta_0}$ into the G -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules on \mathfrak{X}_∞ . As in [18] we tentatively call its essential image the coadmissible (equivariant) $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules. We then obtain

Theorem 2. *The functors $\mathcal{L}oc_\infty^\dagger$ and $H^0(\mathfrak{X}_\infty, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G)_{\theta_0}$ -modules and coadmissible equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules.*

When the projective system \mathcal{F} is cofinal in the system of all formal models of X^{rig} , then $\mathfrak{X}_\infty = \mathfrak{X}_\infty(\mathcal{F})$ is the Zariski-Riemann space attached to X^{rig} . The latter space is in turn

¹considered as a locally L -analytic group

isomorphic (as a ringed space, after inverting p on the structure sheaf) to the adic space attached to X^{rig} , cf. [19].

With Theorem 1 as key ingredient, the proof of Theorem 2 is a straightforward generalization of the GL_2 -case treated in [18], but we give all the details.

In this paper we only treat the case of the central character θ_0 , but there is an extension of this theorem available for characters more general than θ_0 by using twisted versions of the sheaves $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. Moreover, the construction of the sheaf $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ carries over to general smooth rigid (or adic) spaces over L . We will discuss these addenda in a sequel paper.

We would also like to mention that K. Ardakov and S. Wadsley are developing a theory of \mathcal{D} -modules on general rigid spaces, cf. [1], [2]. In their work they consider deformations of the sheaves of crystalline differential operators (as in [3]), whereas we take as a starting point deformations of Berthelot's rings of arithmetic differential operators. Insight from prior studies of logarithmic arithmetic differential operators (cf. [17], [16]) suggested to consider these latter deformations, because their global sections turned out to be (central reductions of) the analytic distribution algebras mentioned above.

Notation. L denotes a finite extension of \mathbb{Q}_p , with ring of integers \mathfrak{o} and uniformizer ϖ . We put $\mathfrak{S} = \text{Spf}(\mathfrak{o})$ and $S = \text{Spec}(\mathfrak{o})$. Let q denote the cardinality of the residue field $\mathfrak{o}/(\varpi)$ which we also denote by \mathbb{F}_q . \mathbb{G}_0 denotes a split reductive group scheme over \mathfrak{o} and $\mathbb{B}_0 \subset \mathbb{G}_0$ a Borel subgroup scheme. The Lie algebra of \mathbb{G}_0 is denoted by \mathfrak{g}_0 . For any scheme X over \mathfrak{o} , we denote by \mathcal{T}_X its relative tangent sheaf. A coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ is called *open* if it contains a power $\varpi^N \mathcal{O}_X$. A scheme which arises from blowing up an open ideal sheaf on X will be called *an admissible blow-up* of X . We always denote by \mathfrak{X} the completion of X along its special fiber.

2. THE SHEAVES $\tilde{\mathcal{D}}_{X,k}^{(m)}$ AND $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$

2.1. Definitions. In this section, X_0 denotes a smooth S -scheme, and \mathfrak{X}_0 its formal completion. Let $\text{pr} : X \rightarrow X_0$ be an admissible blow-up of the scheme X_0 , defined by a sheaf of ideals \mathcal{I} , containing ϖ^N and \mathfrak{X} be the formal completion of X . For any integer $k \geq N$, and m a fixed positive integer, we will define a p -adically complete sheaf of arithmetic differential operators $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ over the non-smooth formal scheme \mathfrak{X} . To achieve this construction, we will first define a sheaf of differential operators $\tilde{\mathcal{D}}_{X,k}^{(m)}$ on X for every $k \geq N$.

The scheme X_0 being a smooth S -scheme, one can use the sheaves of arithmetic differential operators of Berthelot defined in [5]. In particular, for a fixed $m \in \mathbb{N}$, $\mathcal{D}_{X_0}^{(m)}$ will denote the sheaf of differential operators over X_0 of level m . The usual sheaf of differential operators (EGA IV) over X_0 will be denoted by \mathcal{D}_{X_0} .

Let U be a smooth affine open scheme of X_0 endowed with coordinates x_1, \dots, x_N and \mathfrak{U} its formal completion. One denotes ∂_l the derivation relative to x_l , $\partial_l^{[n]} \in \mathcal{D}_U$ defined by

$\nu_i! \partial_l^{[\nu_i]} = \partial_l^{\nu_i}$, and finally

$$\partial_l^{\langle \nu_i \rangle} = q_{\nu_i}! \partial_l^{[\nu_i]}.$$

Here, q_{ν_i} denotes the quotient of the euclidean division of ν_i by p^m . For $(\nu_1, \dots, \nu_N) \in \mathbb{N}^N$, let us define $\underline{\partial}^{\langle \underline{\nu} \rangle} = \prod_{l=1}^N \partial_l^{\langle \nu_l \rangle}$, and $\underline{\partial}^{[\underline{\nu}]} = \prod_{l=1}^N \partial_l^{[\nu_l]}$. Restricted to U , the sheaf $\mathcal{D}_{X_0, k}^{(m)}$ is a sheaf of free \mathcal{O}_U -algebras, with basis the elements $\underline{\partial}^{\langle \underline{\nu} \rangle}$.

Following [18], given two positive integers, k, m , one introduces the subsheaves of algebras $\tilde{\mathcal{D}}_{X_0, k}^{(m)}$ (resp. $\tilde{\mathcal{D}}_{\mathfrak{X}, k}^{(m)}$) of differential operators of congruence level k , which are locally free \mathcal{O}_{U_i} -algebras (resp. $\mathcal{O}_{\mathcal{U}}$ -algebras), with basis the elements $\varpi^{k|\underline{\nu}|} \underline{\partial}^{\langle \underline{\nu} \rangle}$, meaning that

$$\tilde{\mathcal{D}}_{X_0, k}^{(m)}(U) = \left\{ \sum_{\underline{\nu}} \varpi^{k|\underline{\nu}|} a_{\underline{\nu}} \underline{\partial}^{\langle \underline{\nu} \rangle} \mid a_{\underline{\nu}} \in \mathcal{O}_{X_0}(U) \right\}.$$

Let m be a fixed non negative integer, if ν is a non negative integer, one denotes by q the quotient of the euclidean division of ν by p^m . Let $\nu \geq \nu'$ be two nonnegative integers and $\nu'' := \nu - \nu'$, then we introduce for the corresponding numbers q, q' and q'' the following notation

$$\left\{ \begin{matrix} \nu \\ \nu' \end{matrix} \right\} = \frac{q!}{q'! q''!}.$$

We also define

$$\tilde{\mathcal{D}}_{\mathfrak{X}_0, k}^{(m)} = \varprojlim_i \tilde{\mathcal{D}}_{X_0, k}^{(m)} / p^i \quad \text{and} \quad \tilde{\mathcal{D}}_{\mathfrak{X}_0, k, \mathbb{Q}}^{\dagger} = \varinjlim_m \tilde{\mathcal{D}}_{\mathfrak{X}_0, k}^{(m)} \otimes \mathbb{Q}.$$

As explained in [18], these sheaves of rings have the same finiteness properties as the usual sheaves of Berthelot (corresponding to $k = 0$). In particular, over an affine smooth scheme U_0 there is an equivalence of categories between the coherent $\tilde{\mathcal{D}}_{U_0, k}^{(m)}$ -modules and the finite type modules over the algebra $\Gamma(U_0, \tilde{\mathcal{D}}_{U_0, k}^{(m)})$, for every $k \geq 0$.

Let us now explain how to construct the desired sheaves over X . Imitating 2.3.5 of [5], if we can prove that \mathcal{O}_X can be endowed with a structure of $\mathrm{pr}^{-1} \tilde{\mathcal{D}}_{X_0, k}^{(m)}$ -module, then the sheaf of \mathcal{O}_X -modules, $\mathrm{pr}^* \tilde{\mathcal{D}}_{X_0, k}^{(m)}$, will be a sheaf of rings over \mathcal{O}_X . In order to prove this, we need the following proposition.

Proposition 2.1.1. *Let $k \geq N$. Then the sheaf of rings \mathcal{O}_X can be endowed with a structure of $\mathrm{pr}^{-1} \tilde{\mathcal{D}}_{X_0, k}^{(m)}$ -module, such that the map $\mathrm{pr}^{-1} \mathcal{O}_{X_0} \rightarrow \mathcal{O}_X$ is $\mathrm{pr}^{-1} \tilde{\mathcal{D}}_{X_0, k}^{(m)}$ -linear.*

Proof. It is enough to prove the statement in the case where X_0 is affine, say $X_0 = \mathrm{Spec} A$. Denote then $I = \Gamma(X_0, \mathcal{I})$, which contains ϖ^N , and B the A -graded algebra $B = \mathrm{Sym}_A I$, $B = \bigoplus_n I_n$ (meaning that I_n equals as an abelian group I^n), so that $X = \mathrm{Proj} B$. Let $t \in I_d$ an homogeneous element of degree $d > 0$ of B , and $B[1/t]_0$ the algebra of degree 0-elements in the localization $B[1/t]$. It is enough to prove that $B[1/t]_0$ can be endowed with a structure of $\tilde{\mathcal{D}}_{X_0, k}^{(m)}(X_0)$ -module. We can even assume that X_0 is affine endowed

with local coordinates x_1, \dots, x_N . In this case, we let $D_k^{(m)} = \Gamma(X_0, \tilde{\mathcal{D}}_{X_0, k}^{(m)})$ which can be described in the following way

$$D_k^{(m)} = \left\{ \sum_{\underline{\nu}} \varpi^{k|\underline{\nu}|} a_{\underline{\nu}} \partial^{\langle \underline{\nu} \rangle} \mid a_{\underline{\nu}} \in A \right\}.$$

Let us observe that the algebra B is a subgraded algebra of $A[T]$. Indeed, there is a graded ring morphism

$$B \xrightarrow{\varphi} A[T]$$

$$x_n \in I_n \longrightarrow x_n T^n.$$

We can identify $\text{Spec } A[T]$ with $Y = \mathbb{A}_{X_0}^1$ and consider $\text{pr}_2: Y \rightarrow X_0$. By copying the classical proof, one can check that $\text{pr}_2^{-1} \tilde{\mathcal{D}}_{X_0, k}^{(m)}$ is a subsheaf of the sheaf $\tilde{\mathcal{D}}_{Y, k}^{(m)}$, as sheaf of rings. Let U_t be the affine open set of Y where $\varphi(t)$ is nonzero. and $C_t = \Gamma(U_t, \mathcal{O}_Y)$. By flatness of localization, there is an injective map $B_0[1/t] \subset C_t$. Moreover, C_t and $A[T]$ are $\Gamma(Y, \tilde{\mathcal{D}}_{Y, k}^{(m)})$ -modules and thus $D_k^{(m)}$ -modules. Take $P \in D_k^{(m)}$ and $a \in A$, then $P(aT^n) = P(a)T^n$. This shows that the action of $D_k^{(m)}$ over $A[T]$ is graded.

To prove that $B_0[1/t]$ is a $D_k^{(m)}$ -module, we will first prove that B is a $D_k^{(m)}$ -submodule of $A[T]$ and then that $B_0[1/t]$ is a $D_k^{(m)}$ -submodule of C_t . The fact that the defined actions will be compatible with the action of $D_k^{(m)}$ over A , will come from the fact that this is true for the action of $D_k^{(m)}$ over $A[T]$.

Let us prove now that B is a $D_k^{(m)}$ -submodule of $A[T]$. To avoid too heavy notations, we identify B (resp. B_t) with their image into $A[T]$ (resp. C_t) via φ . The ideal I is trivially stable by the action of A by left multiplication, and $A = I_0$ is certainly a $D_k^{(m)}$ -module. Recall that the algebra $D_k^{(m)}$ is generated by the operators $\varpi^{k\nu_i} \partial_i^{\langle \nu_i \rangle}$ for all $\nu_i \leq p^m$. Since $k \geq N$, it is clear that if $\nu_i \geq 1$, $\varpi^{k\nu_i} \partial_i^{\langle \nu_i \rangle}(IT) \subset \varpi^N AT \subset IT$, since the action of $D_k^{(m)}$ over $A[T]$ is graded, so that we can define an action of $D_k^{(m)} : I_1 \rightarrow I_1$.

We next prove by induction on d that I_d is stable by the action of $D_k^{(m)}$. Let

$$x = \varpi^{N(d-b)} y_1 \dots y_b \in I_d.$$

Let $\nu_i \geq 1$ and $\nu_i \leq p^m$, then one computes

$$\varpi^{N\nu_i} \partial_i^{\langle \nu_i \rangle}(y_1 \dots y_b) = \sum_{\mu_i \leq \nu_i} \left(\varpi^{N\mu_i} \left\{ \begin{matrix} \mu_i \\ \nu_i \end{matrix} \right\} \partial_i^{\langle \mu_i \rangle}(y_1) \right) \varpi^{N(\nu_i - \mu_i)} \partial_i^{\langle \nu_i - \mu_i \rangle}(y_2 \dots y_b)$$

by [5, 2.2.4(iv)]. By induction, we know that $\varpi^{N(\nu_i - \mu_i)} \partial_i^{\langle \nu_i - \mu_i \rangle}(y_2 \dots y_b) \in I_{b-1}$, that implies that

$$\varpi^{N\nu_i} \partial_i^{\langle \nu_i \rangle}(x) \in I_d.$$

It remains now to prove that $B[1/t]_0$ is a $D_k^{(m)}$ -submodule of C_t . We have to check the following statement

$$(2.1.1) \quad \forall \nu_i \leq p^m, \forall c \in \mathbb{N}, \forall g \in I_{dc}, \varpi^{N\nu_i} \partial_i^{\langle \nu_i \rangle} \left(\frac{g}{t^c} \right) \in B_0[1/t].$$

Let us first prove by induction on ν_i that

$$\varpi^{N\nu_i} \partial_i^{\langle \nu_i \rangle} (t^{-1}) \in \frac{I_{\nu_i d}}{t^{\nu_i+1}}.$$

This is true for $\nu_i = 0$. Consider then the formula

$$\varpi^{N(\nu_i+1)} \partial_i^{\langle \nu_i+1 \rangle} (t^{-1}) = - \sum_{\mu=0}^{\nu_i} \left\{ \begin{matrix} \nu_i+1 \\ \mu \end{matrix} \right\} t^{-1} \varpi^{N(\nu_i+1-\mu)} \partial_i^{\langle \nu_i+1-\mu \rangle} (t) \varpi^{N\mu} \partial_i^{\langle \mu \rangle} (t^{-1}).$$

By induction on ν_i , one knows that for any integer $\mu \leq \nu_i$,

$$\varpi^{N(\nu_i+1)} t^{-1} \partial_i^{\langle \nu_i+1-\mu \rangle} (t) \partial_i^{\langle \mu \rangle} (t^{-1}) \in \frac{1}{t^{\mu+2}} I_d I_{\mu d} \subset \frac{I_{(\nu_i+1)d}}{t^{\nu_i+2}},$$

which proves our claim. Applying this claim to t^c gives for $\mu \leq p^m$

$$\varpi^{N\mu} \partial_i^{\langle \mu \rangle} (t^{-c}) \in \frac{I_{\mu dc}}{t^{c(\mu+1)}}.$$

Then, we have the following formula

$$\varpi^{N\nu_i} \partial_i^{\langle \nu_i \rangle} \left(\frac{g}{t^c} \right) = \sum_{\mu=0}^{\nu_i} \left\{ \begin{matrix} \nu_i \\ \mu \end{matrix} \right\} \varpi^{N(\nu_i-\mu)} \partial_i^{\langle \nu_i-\mu \rangle} (g) \varpi^{N\mu} \partial_i^{\langle \mu \rangle} (t^{-c}),$$

whose right-hand terms are contained in

$$I_{dc} \frac{I_{\mu dc}}{t^{c(\mu+1)}} \subset B_0[1/t],$$

and this completes the proof of 2.1.1 and the proposition. \square

From this, we deduce as in 2.3.5 of [5] the following.

Corollary 2.1.2. *Under the hypothesis of the section, for $k \geq N$, the sheaf $\mathrm{pr}^* \tilde{\mathcal{D}}_{X_0,k}^{(m)}$ is a sheaf of rings over X .*

This allows us to introduce the following sheaves of rings over the admissible blow-up \mathfrak{X} of \mathfrak{X}_0

$$\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)} = \varprojlim_i \mathrm{pr}^* \tilde{\mathcal{D}}_{X_0,k}^{(m)} / p^i \quad \text{and} \quad \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger = \varinjlim_m \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)} \otimes \mathbb{Q}.$$

We abbreviate $\tilde{\mathcal{D}}_{X,k}^{(m)} := \mathrm{pr}^* \tilde{\mathcal{D}}_{X_0,k}^{(m)}$ in the following.

2.2. Finiteness properties of the sheaves $\tilde{\mathcal{D}}_{X,k}^{(m)}$. We keep here the hypothesis of the beginning of the section. For a given natural number $k \geq 0$ we let

$$\tilde{\mathcal{T}}_{X,k} := \varpi^k(\mathrm{pr})^*(\mathcal{T}_{X_0}).$$

Lemma 2.2.1. (i) $\tilde{\mathcal{T}}_{X,k}$ is a locally free \mathcal{O}_X -module of rank equal to the relative dimension of X_0 over S ,

(ii) Suppose $\pi : X' \rightarrow X$ is a morphism of admissible blow-ups of X_0 . Let $k', k \geq 0$. One has

$$\tilde{\mathcal{T}}_{X',k'} = \varpi^{k'-k} \pi^*(\tilde{\mathcal{T}}_{X,k})$$

as subsheaves of $\mathcal{T}_{X'} \otimes L$.

Proof. The assertion (i) follows directly from the definition. The assertion (ii) follows since π is a morphism over X_0 . \square

We also have

Proposition 2.2.2. (i) The sheaves $\tilde{\mathcal{D}}_{X,k}^{(m)}$ are filtered by the order of differential operators and there is a canonical isomorphism of graded sheaves of algebras

$$\mathrm{gr} \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right) \simeq \mathrm{Sym}(\tilde{\mathcal{T}}_{X,k})^{(m)} = \bigoplus_{d \geq 0} \mathrm{Sym}^d(\tilde{\mathcal{T}}_{X,k})^{(m)}.$$

(ii) There is a basis of the topology \mathcal{B} of X , consisting of open affine subsets, such that for any $U \in \mathcal{B}$, the ring $\tilde{\mathcal{D}}_{X,k}^{(m)}(U)$ is noetherian. In particular, the sheaf of rings $\tilde{\mathcal{D}}_{X,k}^{(m)}$ is coherent.

(iii) The sheaf $\tilde{\mathcal{D}}_{X,k}^{(m)}$ is coherent.

Proof. Denote by $\tilde{\mathcal{D}}_{X_0,k;d}^{(m)}$ the sheaf of differential operators of $\tilde{\mathcal{D}}_{X_0,k}^{(m)}$ of order less than d . It is straightforward that we have an exact sequence of \mathcal{O}_{X_0} -modules on X_0

$$(2.2.3) \quad 0 \longrightarrow \tilde{\mathcal{D}}_{X_0,k;d-1}^{(m)} \longrightarrow \tilde{\mathcal{D}}_{X_0,k;d}^{(m)} \longrightarrow \mathrm{Sym}^d(\tilde{\mathcal{T}}_{X_0,k})^{(m)} \longrightarrow 0.$$

Now we apply pr^* and get an exact sequence since $\mathrm{Sym}^d(\tilde{\mathcal{T}}_{X_0,k})^{(m)}$ is a free \mathcal{O}_{X_0} -module of finite rank. This gives (i). As we work with quasi-coherent sheaves of \mathcal{O}_X -modules, it is enough to show that $\tilde{\mathcal{D}}_{X,k}^{(m)}(U)$ is noetherian in the case where $U = \mathrm{pr}^{-1}(U_0)$, and $U_0 \subset X_0$ has some coordinates x_1, \dots, x_N . In this case one has the following description

$$\tilde{\mathcal{D}}_{X,k}^{(m)}(U) = \left\{ \sum_{\underline{\nu}} \varpi^{k|\underline{\nu}|} a_{\underline{\nu}} \partial^{\langle \underline{\nu} \rangle} \mid a_{\underline{\nu}} \in \mathcal{O}_X(U) \right\}.$$

By (i), the graded algebra $\mathrm{gr} \tilde{\mathcal{D}}_{X,k}^{(m)}(U)$ is isomorphic to $\mathrm{Sym}_{\mathcal{O}(U)}(\tilde{\mathcal{T}}_{X,k}(U))^{(m)}$, which is known to be noetherian [13, Prop. 1.3.6]. This gives the noetherianity of the algebras

$\tilde{\mathcal{D}}_{X,k}^{(m)}(U)$. As \mathcal{B} we may take the set of open subsets of X that are contained in some $\mathrm{pr}^{-1}(U_0)$, for some open $U_0 \subset X_0$ endowed with global coordinates. We finally note that, since the sheaf $\tilde{\mathcal{D}}_{X,k}^{(m)}$ is \mathcal{O}_X -quasicoherent and has noetherian sections of over open affines, it is actually a sheaf of coherent rings [5, 3.1.3(i)]. The last assertion is a direct consequence of (ii), as in [5]. We do not redo the proof here. \square

From these considerations, we give theorems A and B for affine schemes X , whose proofs follow readily by the proofs given by Berthelot in [5].

Proposition 2.2.4. *Let $U \subset X$ be an affine subscheme of X , \mathfrak{U} its formal completion.*

- (i) *The algebra $\Gamma(U, \tilde{\mathcal{D}}_{U,k}^{(m)})$ is noetherian and the functor $\Gamma(U, \cdot)$ establishes an equivalence of categories between coherent $\tilde{\mathcal{D}}_{U,k}^{(m)}$ -modules and finite $\Gamma(U, \tilde{\mathcal{D}}_{U,k}^{(m)})$ -modules.*
- (ii) *The algebra $\Gamma(\mathfrak{U}, \tilde{\mathcal{D}}_{\mathfrak{U},k}^{(m)})$ is noetherian and the functor $\Gamma(\mathfrak{U}, \cdot)$ establishes an equivalence of categories between coherent $\tilde{\mathcal{D}}_{\mathfrak{U},k}^{(m)}$ -modules and finite $\Gamma(\mathfrak{U}, \tilde{\mathcal{D}}_{\mathfrak{U},k}^{(m)})$ -modules.*

3. FORMAL MODELS OF FLAG VARIETIES

From now on X_0 denotes the smooth flag scheme $\mathbb{G}_0/\mathbb{B}_0$ of \mathbb{G}_0 and \mathfrak{X}_0 denotes its p -adic completion.

3.1. Congruence group schemes. We let $\mathbb{G}(k)$ denote the k -th scheme-theoretic congruence subgroup of the group scheme \mathbb{G}_0 [20, 1.], [21, 2.8]. So $\mathbb{G}(0) = \mathbb{G}_0$ and $\mathbb{G}(k+1)$ equals the dilatation (in the sense of [7, 3.2]) of the trivial subgroup of $\mathbb{G}(k) \otimes_{\mathbb{F}_q} \mathbb{F}_q$ on $\mathbb{G}(k)$. In particular, if $\mathbb{G}(k) = \mathrm{Spec} \, \mathfrak{o}[t_1, \dots, t_N]$ with a set of parameters t_i for the unit section of $\mathbb{G}(k)$, then $\mathbb{G}(k+1) = \mathrm{Spec} \, \mathfrak{o}[\frac{t_1}{\varpi}, \dots, \frac{t_N}{\varpi}]$. The \mathfrak{o} -group scheme $\mathbb{G}(k)$ is again smooth, has Lie algebra equal to $\varpi^k \mathfrak{g}_{\mathfrak{o}}$ and its generic fibre coincides with the generic fibre of \mathbb{G}_0 .

3.2. A very ample line bundle on X . Let $\mathrm{pr} : X \rightarrow X_0$ be an admissible blow-up, and let $\mathcal{I} \subset \mathcal{O}_{X_0}$ be the ideal sheaf that is blown up. Since \mathcal{I} is open, there is $N \in \mathbb{Z}_{>0}$ such that

$$(3.2.1) \quad p^N \mathcal{O}_{X_0} \subset \mathcal{I} \subset \mathcal{O}_{X_0}.$$

Put $\mathcal{S} = \bigoplus_{s \geq 0} \mathcal{I}^s$, then X is glued together from schemes $\mathbf{Proj}(\mathcal{S}(U))$ for affine open subsets $U \subset X$. On each $\mathbf{Proj}(\mathcal{S}(U))$ there is an invertible sheaf $\mathcal{O}(1)$, and these glue together to give an invertible sheaf $\mathcal{O}(1)$ on X which we will denote $\mathcal{O}_{X/X_0}(1)$ (cf. the discussion in [12, ch. II, §7].) This invertible sheaf is in fact the inverse image ideal sheaf $\mathrm{pr}^{-1}(\mathcal{I}) \cdot \mathcal{O}_X$, cf. [12, ch. II, 7.13]. From 3.2.1 conclude that

$$(3.2.2) \quad p^N \mathcal{O}_X \subset \mathcal{O}_{X/X_0}(1) \subset \mathcal{O}_X \quad \text{and} \quad \mathcal{O}_X \subset \mathcal{O}_{X/X_0}(-1) \subset p^{-N} \mathcal{O}_X.$$

And for any $r \geq 0$ we get

$$(3.2.3) \quad p^{rN} \mathcal{O}_X \subset \mathcal{O}_{X/X_0}(1)^{\otimes r} \subset \mathcal{O}_X \quad \text{and} \quad \mathcal{O}_X \subset \mathcal{O}_{X/X_0}(-1)^{\otimes r} \subset p^{-rN} \mathcal{O}_X.$$

Lemma 3.2.4. *Suppose X is normal. Let X' be the blow-up of a coherent open ideal on X and let $\text{pr} : X' \rightarrow X$ be the blow-up morphism. Then $(\text{pr})_*(\mathcal{O}_{X'}) = \mathcal{O}_X$.*

Proof. The morphism $\text{pr} : X' \rightarrow X$ is a birational projective morphism of noetherian integral schemes. The assertion then follows exactly as in the proof of Zariski's Main Theorem as given in [12, ch. III, Cor. 11.4]. \square

Lemma 3.2.5. *There is $a_0 \in \mathbb{Z}_{>0}$ such that the line bundle*

$$(3.2.6) \quad \mathcal{L}_X = \mathcal{O}_{X/X_0}(1) \otimes \text{pr}^*(\mathcal{O}_{X_0}(a_0))$$

on X is very ample over $\text{Spec}(\mathfrak{o})$, and it is very ample over X_0 .

Proof. By [12, ch. II, ex. 7.14 (b)], the sheaf

$$(3.2.7) \quad \mathcal{L} = \mathcal{O}_{X/X_0}(1) \otimes \text{pr}^*(\mathcal{O}_{X_0}(a_0))$$

is very ample on X over $\text{Spec}(\mathfrak{o})$ for suitable $a_0 > 0$. We fix such an a_0 . By [11, 4.4.10 (v)] it is then also very ample over X_0 . \square

Lemma 3.2.8. *Suppose $\pi : X' \rightarrow X$ is a morphism of admissible blow-ups of X_0 .*

(i) In the case $\pi_ \mathcal{O}_{X'} = \mathcal{O}_X$, one has*

$$\tilde{\mathcal{T}}_{X,k} = \varpi^{k-k'} \pi_*(\tilde{\mathcal{T}}_{X',k'})$$

as subsheaves of $\mathcal{T}_X \otimes L$.

(ii) There is a \mathcal{O}_X -linear surjection $\mathcal{O}_X \otimes_{\mathfrak{o}} \varpi^k \mathfrak{g}_{\mathfrak{o}} \twoheadrightarrow \tilde{\mathcal{T}}_{X,k}$.

Proof. The assertion (i) follows from (ii) of the previous lemma 2.2.1 together with the projection formula. Finally, (ii) follows from [15, 1.6.1] by applying π^* . \square

Proposition 3.2.9. *In the case $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$, one has $\pi_* \left(\tilde{\mathcal{D}}_{X',k}^{(m)} \right) = \tilde{\mathcal{D}}_{X,k}^{(m)}$.*

Proof. The sheaves $\tilde{\mathcal{D}}_{X_0,k;d}^{(m)}$ are locally free of finite rank, and so are the sheaves $\tilde{\mathcal{D}}_{X,k;d}^{(m)}$, by construction. We can thus apply the projection formula and get

$$\pi_* \left(\tilde{\mathcal{D}}_{X',k;d}^{(m)} \right) = \tilde{\mathcal{D}}_{X,k;d}^{(m)} .$$

The claim follows because the direct image commutes with inductive limits on a noetherian space. \square

3.2.10. Twisting by \mathcal{L}_X . Recall the very ample line bundle \mathcal{L}_X from 3.2.5. In the following we will always use this line bundle to 'twist' \mathcal{O}_X -modules. If \mathcal{F} is a \mathcal{O}_X -module and $r \in \mathbb{Z}$ we thus put

$$\mathcal{F}(r) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X^{\otimes r} .$$

Some caveat is in order when we deal with sheaves which are equipped with both a left and a right \mathcal{O}_X -module structure (which may not coincide). For instance, if $\mathcal{F}_d = \tilde{\mathcal{D}}_{X,k;d}^{(m)}$ then we let

$$\mathcal{F}_d(r) = \tilde{\mathcal{D}}_{X,k;d}^{(m)}(r) = \tilde{\mathcal{D}}_{X,k;d}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{L}_X^{\otimes r} ,$$

where we consider $\mathcal{F}_d = \tilde{\mathcal{D}}_{X,k;d}^{(m)}$ as a *right* \mathcal{O}_X -module. Similarly we put

$$\tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{L}_X^{\otimes r} ,$$

where we consider $\tilde{\mathcal{D}}_{X,k}^{(m)}$ as a *right* \mathcal{O}_X -module. Then we have $\tilde{\mathcal{D}}_{X,k}^{(m)}(r) = \varinjlim_d \mathcal{F}_d(r)$. When we consider the associated graded sheaf of $\tilde{\mathcal{D}}_{X,k}^{(m)}(r)$, it is with respect to the filtration by the $\mathcal{F}_d(r)$. The sheaf $\tilde{\mathcal{D}}_{X,k}^{(m)}(r)$ is a coherent left $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module according to part (iii) of 3.2.9.

3.3. Global sections of $\tilde{\mathcal{D}}_{X,k}^{(m)}$, $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$, and $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$.

3.3.1. Divided power enveloping algebras. We denote by $D^{(m)}(\mathbb{G}(k))$ the distribution algebra of the smooth \mathfrak{o} -group scheme $\mathbb{G}(k)$ of level m [14]. It is noetherian and admits the following explicit description. Let $\mathfrak{g}_{\mathfrak{o}} = \mathfrak{n}_{\mathfrak{o}}^- \oplus \mathfrak{t}_{\mathfrak{o}} \oplus \mathfrak{n}_{\mathfrak{o}}$ be a triangular decomposition of $\mathfrak{g}_{\mathfrak{o}}$. We fix basis elements $(f_i), (h_j)$ and (e_i) of the \mathfrak{o} -modules $\mathfrak{n}_{\mathfrak{o}}^-, \mathfrak{t}_{\mathfrak{o}}$ and $\mathfrak{n}_{\mathfrak{o}}$ respectively. Then $D^{(m)}(\mathbb{G}(k))$ equals the \mathfrak{o} -subalgebra of $U(\mathfrak{g}) = U(\mathfrak{g}_{\mathfrak{o}}) \otimes_{\mathfrak{o}} L$ generated by the elements

$$(3.3.2) \quad q_{\underline{\nu}}^{(m)}! \frac{(\varpi^k \underline{e})^{\underline{\nu}}}{\underline{\nu}!} \cdot q_{\underline{\nu}'}^{(m)}! \varpi^{k|\underline{\nu}'|} \left(\frac{\underline{h}}{\underline{\nu}'} \right) \cdot q_{\underline{\nu}''}^{(m)}! \frac{(\varpi^k \underline{f})^{\underline{\nu}''}}{\underline{\nu}''!} .$$

In the case of the group GL_2 we considered the same algebra in [18]. We denote by $\hat{D}^{(m)}(\mathbb{G}(k))$ the p -adic completion of $D^{(m)}(\mathbb{G}(k))$. It is noetherian.

Proposition 3.3.3. *There is a canonical homomorphism of \mathfrak{o} -algebras*

$$(3.3.4) \quad Q_{X,k}^{(m)} : D^{(m)}(\mathbb{G}(k)) \rightarrow H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)}),$$

Proof. Recall that a filtered \mathfrak{o} -algebra (or sheaf of such algebras) A with positive filtration $F_i A$ and $\mathfrak{o} \subseteq F_0 A$ yields the graded subring $R(A) := \bigoplus_{i \geq 0} F_i A t^i \subseteq A[t]$ of the polynomial ring over A , its associated Rees ring. Specialising $R(A)$ in an element $\lambda \in \mathfrak{o}$ yields a filtered subring A_λ of A . For fixed λ , the formation of A_λ is functorial in A . We apply this remark to the filtered homomorphism

$$D^{(m)}(\mathbb{G}(0)) \rightarrow \tilde{\mathcal{D}}_{X_0,0}^{(m)}$$

appearing in [14, 4.4.4] (and denoted by Q_m in loc.cit.) which comes by functoriality from the $\mathbb{G}(0)$ -action on X_0 . Passing to Rees rings and specialising the parameter in ϖ^k yields a filtered homomorphism

$$D^{(m)}(\mathbb{G}(k)) \rightarrow \tilde{\mathcal{D}}_{X_0,k}^{(m)}.$$

Taking global sections and using $H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)}) = H^0(X_0, \tilde{\mathcal{D}}_{X_0,k}^{(m)})$ by 3.2.9 yields a homomorphism

$$Q_{X,k}^{(m)} : D^{(m)}(\mathbb{G}(k)) \rightarrow H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)})$$

as claimed. \square

Let $\mathcal{A}_{X,k}^{(m)} = \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k))$ with the twisted ring multiplication coming from the action of $D^{(m)}(\mathbb{G}(k))$ on \mathcal{O}_X via $Q_{X,k}^{(m)}$. It has a natural filtration whose associated graded equals the \mathcal{O}_X -algebra $\mathcal{O}_X \otimes_{\mathfrak{o}} \text{Sym}^{(m)}(\text{Lie}(\mathbb{G}(k)))$ [14, Cor. 4.4.6]. In particular, $\mathcal{A}_{X,k}^{(m)}$ has noetherian sections over open affines.

Proposition 3.3.5. *The homomorphism $\mathcal{A}_{X,k}^{(m)} \rightarrow \tilde{\mathcal{D}}_{X,k}^{(m)}$ induced by $\xi_{X,k}^{(m)}$ is surjective. In particular, $\tilde{\mathcal{D}}_{X,k}^{(m)}$ has noetherian sections over open affines.*

Proof. We adapt the argument of [14, 4.4.7.2(ii)]: The homomorphism is filtered. Applying $\text{Sym}^{(m)}$ to the surjection in (iv) of 3.2.8 we obtain a surjection $\mathcal{O}_X \otimes_{\mathfrak{o}} \text{Sym}^{(m)}(\text{Lie}(\mathbb{G}(k))) \rightarrow \text{Sym}^{(m)}(\tilde{\mathcal{T}}_{X,k})$ which equals the associated graded homomorphism by 3.2.9. Hence the homomorphism is surjective as claimed. \square

Proposition 3.3.6. *Let \mathcal{M} be a coherent $\mathcal{A}_{X,k}^{(m)}$ -module.*

(i) $H^0(X, \mathcal{A}_{X,k}^{(m)}) = D^{(m)}(\mathbb{G}(k))$.

(ii) *There is a surjection $\mathcal{A}_{X,k}^{(m)}(-r)^{\oplus s} \rightarrow \mathcal{M}$ of $\mathcal{A}_{X,k}^{(m)}$ -modules for suitable $r, s \geq 0$.*

(iii) *For any $i \geq 0$ the group $H^i(X, \mathcal{M})$ is a finitely generated $D^{(m)}(\mathbb{G}(k))$ -module.*

(iv) The ring $H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)})$ is a finitely generated $D^{(m)}(\mathbb{G}(k))$ -module and hence noetherian.

Proof. The points (i)-(iii) follow exactly as in [14, A.2.6.1]. Statement (iv) is a special case of (iii) by 3.3.5. \square

Now let $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ be the p -adic completion of $\tilde{\mathcal{D}}_{X,k}^{(m)}$, which we will always consider as a sheaf on the formal scheme \mathfrak{X} . We also put $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)} = \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)} \otimes_{\mathbb{Z}} \mathbb{Q}$, and

$$(3.3.7) \quad \tilde{\mathcal{D}}_{\mathfrak{X},k}^{\dagger} = \varinjlim_m \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)},$$

and

$$(3.3.8) \quad \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger} = \tilde{\mathcal{D}}_{\mathfrak{X},k}^{\dagger} \otimes_{\mathbb{Z}} \mathbb{Q} = \varinjlim_m \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}.$$

We denote by $\hat{D}^{(m)}(\mathbb{G}(k))_{\mathbb{Q},\theta_0}$ the quotient of $\hat{D}^{(m)}(\mathbb{G}(k)) \otimes \mathbb{Q}$ modulo the ideal generated by the center of the ring $U(\mathfrak{g})$ [14, A.2.1]. This is the same central reduction considered in [18] for the group GL_2 .

In the proposition below, and in the remainder of this paper, certain rigid analytic ‘wide open’ groups $\mathbb{G}(k)^{\circ}$ will be used repeatedly. To define them, consider first the formal completion $\mathfrak{G}(k)$ of the group scheme $\mathbb{G}(k)$ along its special fiber, which is a formal group scheme (of topologically finite type) over $\mathfrak{S} = \mathrm{Spf}(\mathfrak{o})$. Then let $\hat{\mathfrak{G}}(k)^{\circ}$ be the completion of $\mathfrak{G}(k)$ along its unit section $\mathfrak{S} \rightarrow \mathfrak{G}(k)$, and denote by $\mathbb{G}(k)^{\circ}$ its associated rigid space.

Proposition 3.3.9. (i) *There is a basis of the topology \mathfrak{U} of X , consisting of open affine subsets, such that for any $U \in \mathfrak{U}$, the ring $H^0(U, \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})$ is noetherian.*

(ii) *The transition map $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)} \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m+1)}$ is flat.*

(iii) *The sheaf of rings $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger}$ is coherent.*

(iv) *The homomorphism $Q_{X,k}^{(m)}$ induces an algebra isomorphism*

$$\hat{D}^{(m)}(\mathbb{G}(k))_{\mathbb{Q},\theta_0} \xrightarrow{\simeq} H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}).$$

(v) *The ring $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger})$ is canonically isomorphic to the coherent L -algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\theta_0}$, where $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})$ is the analytic distribution algebra in the sense of Emerton, cf. [10, ch. 5].*

Proof. (i) This follows from [5, 3.3.4(i)] together with 3.3.5.

- (ii) This can be proved as in [5, sec. 3.5].
- (iii) Follows from (i) and (ii).
- (iv) This follows from statement (iv) in 3.3.6 together with [14, Lem. A.3].
- (v) Follows from (iv) and the description of the analytic distribution algebra as given in [10, ch. 5], compare also [14, Prop. 5.2.1]. \square

4. LOCALIZATION ON \mathfrak{X} VIA $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$

The general line of arguments follows fairly closely [15]. As usual, $\text{pr} : X \rightarrow X_0$ denotes an admissible blow-up of X_0 . The number $k \geq 0$ is fixed throughout this section and chosen large enough so that the sheaf of coherent rings $\tilde{\mathcal{D}}_{X,k}^{(m)}$ is defined for all m .

4.1. Cohomology of coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules.

Lemma 4.1.1. *Let \mathcal{E} be an abelian sheaf on X . For all $i > \dim X$ one has $H^i(X, \mathcal{E}) = 0$.*

Proof. Since the space X is noetherian the result follows from Grothendieck's vanishing theorem [12, Thm. 2.7]. \square

Proposition 4.1.2. *There is a natural number r_0 such that for all $r \geq r_0$ and all $i \geq 1$ one has*

$$(4.1.3) \quad H^i \left(X, \text{gr} \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right) (r) \right) = 0 .$$

Proof. Since \mathcal{L}_X is very ample over \mathfrak{o} by 3.2.5, the Serre theorems [12, II.5.17/III.5.2] imply that there is a number u_0 such that for all $u \geq u_0$ the module $\mathcal{O}_X(u)$ is generated by global sections and has no higher cohomology. After this remark we prove the proposition along the lines of [15, Prop. 2.2.1]. There is an \mathcal{O}_{X_0} -linear surjection $(\mathcal{O}_{X_0})^{\oplus a} \rightarrow \mathcal{T}_{X_0}$ for a suitable natural number a . Applying $(\text{pr})^*$ and multiplication by ϖ^k gives an \mathcal{O}_X -linear surjection $(\mathcal{O}_X)^{\oplus a} \simeq \varpi^k(\mathcal{O}_X)^{\oplus a} \rightarrow \tilde{\mathcal{T}}_{X,k}$. By functoriality we get a surjective morphism of algebras

$$\mathcal{C} := \text{Sym}((\mathcal{O}_X)^{\oplus a})^{(m)} \longrightarrow \text{Sym}(\tilde{\mathcal{T}}_{X,k})^{(m)}.$$

The target of this map equals $\text{gr} \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right)$ according to 3.2.9. It therefore suffices to prove the following: given a coherent \mathcal{C} -module \mathcal{E} , there is a number r_0 such that for all $r \geq r_0$ and $i \geq 1$, one has $H^i(X, \mathcal{E}(r)) = 0$. To start with, X is noetherian, hence \mathcal{E} is quasi-coherent and so equals the union over its \mathcal{O}_X -coherent submodules \mathcal{E}_i . Since \mathcal{E} is \mathcal{C} -coherent and \mathcal{C} has noetherian sections over open affines [13, 1.3.6], there is a \mathcal{C} -linear surjection $\mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{E}_i \rightarrow \mathcal{E}$. Choose a number s_0 such that $\mathcal{E}_i(-s_0)$ is generated by global sections. We obtain a \mathcal{O}_X -linear surjection $\mathcal{O}_X(s_0)^{\oplus a_0} \rightarrow \mathcal{E}_i$ for a number a_0 . This yields

a \mathcal{C} -linear surjection

$$\mathcal{C}_0 := \mathcal{C}(s_0)^{\oplus a_0} \longrightarrow \mathcal{E}.$$

The \mathcal{O}_X -module \mathcal{C}_0 is graded and each homogeneous component equals a sum of copies of $\mathcal{O}_X(s_0)$. It follows that $H^i(X, \mathcal{C}_0(r)) = 0$ for all $r \geq u_0 - s_0$ and all $i \geq 1$. The rest of the argument proceeds now as in [15, 2.2.1]. \square

Corollary 4.1.4. *Let r_0 be the number occuring in the preceding proposition. For all $r \geq r_0$ and all $i \geq 1$ one has*

$$(4.1.5) \quad H^i \left(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(r) \right) = 0 .$$

Proof. For $d \geq 0$ we let $\mathcal{F}_d = \tilde{\mathcal{D}}_{X,k;d}^{(m)}$. We consider the exact sequence

$$(4.1.6) \quad 0 \rightarrow \mathcal{F}_{d-1} \rightarrow \mathcal{F}_d \rightarrow \mathrm{gr}_d \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right) \rightarrow 0$$

(where $\mathcal{F}_{-1} := 0$) from which we deduce the exact sequence

$$(4.1.7) \quad 0 \rightarrow \mathcal{F}_{d-1}(r) \rightarrow \mathcal{F}_d(r) \rightarrow \mathrm{gr}_d \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right)(r) \rightarrow 0$$

because tensoring with a line bundle is an exact functor. Since cohomology commutes with direct sums, we have for all $r \geq r_0$ and $i \geq 1$ that

$$H^i(X, \mathrm{gr}_d \left(\tilde{\mathcal{D}}_{X,k}^{(m)} \right)(r)) = 0$$

according to the preceding proposition. Using the sequence 4.1.7 we can then deduce by induction on d that for all $r \geq r_0$ and $i \geq 1$

$$H^i(X, \mathcal{F}_d(r)) = 0 .$$

Because cohomology commutes with inductive limits on a noetherian scheme we obtain the asserted vanishing result. \square

Proposition 4.1.8. *Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module.*

(i) *There is a number $r = r(\mathcal{E}) \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ and an epimorphism of $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules*

$$\left(\tilde{\mathcal{D}}_{X,k}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathcal{E} .$$

(ii) There is $r_1(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geq r_1(\mathcal{E})$ and all $i > 0$

$$H^i(X, \mathcal{E}(r)) = 0 .$$

Proof. (i) As X is a noetherian scheme, \mathcal{E} is the inductive limit of its coherent subsheaves. There is thus a coherent \mathcal{O}_X -submodule $\mathcal{F} \subset \mathcal{E}$ which generates \mathcal{E} as a $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module, i.e., an epimorphism of sheaves

$$\tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\alpha} \mathcal{E} ,$$

where $\tilde{\mathcal{D}}_{X,k}^{(m)}$ is considered with its right \mathcal{O}_X -module structure. Next, there is $r > 0$ such that the sheaf

$$\mathcal{F}(r) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X^{\otimes r}$$

is generated by its global sections. Hence there is $s > 0$ and an epimorphism $\mathcal{O}_X^{\oplus s} \twoheadrightarrow \mathcal{F}(r)$, and thus an epimorphism of \mathcal{O}_X -modules

$$(\mathcal{O}_X(-r))^{\oplus s} \twoheadrightarrow \mathcal{F} .$$

From this morphism we get an epimorphism of $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules

$$\left(\tilde{\mathcal{D}}_{X,k}^{(m)}(-r) \right)^{\oplus s} = \tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{\mathcal{O}_{X_n}} (\mathcal{O}_X(-r))^{\oplus s} \twoheadrightarrow \tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\alpha} \mathcal{E} .$$

(ii) Consider for $i \geq 1$ the following assertion (a_i) : for any coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module \mathcal{E} , there is a number $r_i(\mathcal{E})$ such that for all $r \geq r_i(\mathcal{E})$ and all $i \leq j$ one has $H^j(X, \mathcal{E}(r)) = 0$. For $i > \dim X$ the assertion holds, cf. 4.1.1. Suppose the statement (a_{i+1}) holds. Using (i) we find an epimorphism of $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules

$$\beta : \mathcal{C}_0 := \left(\tilde{\mathcal{D}}_{X,k}^{(m)}(s_0) \right)^{\oplus s} \twoheadrightarrow \mathcal{E}$$

for numbers $s_0 \in \mathbb{Z}$ and $s \geq 0$. By 3.2.9, the kernel $\mathcal{R} = \ker(\beta)$ is a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module. Recall the number r_0 of the preceding corollary. For any $r \geq \max(r_0 - s_0, r_{i+1}(\mathcal{R}))$ we have the exact sequence

$$0 = H^i(X, \mathcal{C}_0(r)) \longrightarrow H^i(X, \mathcal{E}(r)) \longrightarrow H^{i+1}(X, \mathcal{R}(r)) = 0$$

which shows $H^i(X, \mathcal{E}(r)) = 0$ for these r . So we may take as $r_i(\mathcal{E})$ any of these r which is larger than $r_{i+1}(\mathcal{E})$ and obtain the statement (a_i) . In particular, (a_1) holds which proves (ii). \square

Proposition 4.1.9. (i) Fix $r \in \mathbb{Z}$. There is $c_2 = c_2(r) \in \mathbb{Z}_{\geq 0}$ such that for all $i > 0$ the cohomology group $H^i(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(r))$ is annihilated by p^{c_2} .

(ii) Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module. There is $c_3 = c_3(\mathcal{E}) \in \mathbb{Z}_{\geq 0}$ such that for all $i > 0$ the cohomology group $H^i(X, \mathcal{E})$ is annihilated by p^{c_3} .

Proof. (i) Since the blow-up morphism $\text{pr} : X \rightarrow X_0$ becomes an isomorphism over $X_0 \times_{\circ} L$ any coherent module over $\tilde{\mathcal{D}}_{X,k}^{(m)} \otimes \mathbb{Q}$ induces a coherent module over the sheaf of usual differential operators on $X_0 \times_{\circ} L$. By [4] we conclude that the global section functor on X is exact for coherent $\tilde{\mathcal{D}}_{X,k}^{(m)} \otimes \mathbb{Q}$ -modules. In particular, the cohomology group $H^i(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(r))$ is p -torsion. To see that the torsion is bounded, we deduce from 3.3.5 that $\tilde{\mathcal{D}}_{X,k}^{(m)}(r)$ is a coherent module over $\mathcal{A}_{X,k}^{(m)}$. According to 3.3.6, $H^i(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(r))$ is therefore finitely generated over $D^{(m)}(\mathbb{G}(k))$. This implies the claim.

(ii) We consider for any $i \geq 1$ the following assertion (a_i) : for any coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module \mathcal{E} , there is a number $r_i(\mathcal{E})$ such that the groups $H^j(X, \mathcal{E}), i \leq j$ are all annihilated by $p^{r_i(\mathcal{E})}$. For $i > \dim X$ the assertion holds, cf. 4.1.1. Let us assume (a_{i+1}) holds and consider an arbitrary coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module \mathcal{E} . According to 4.1.8 we have a $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -linear surjection

$$\mathcal{E}_0 := \tilde{\mathcal{D}}_{X,k}^{(m)}(r)^{\oplus s} \longrightarrow \mathcal{E}$$

for numbers $r \in \mathbb{Z}$ and $s \geq 0$. Let \mathcal{E}' be the kernel. We have an exact sequence

$$H^i(X, \mathcal{E}_0) \xrightarrow{\iota} H^i(X, \mathcal{E}) \xrightarrow{\delta} H^{i+1}(X, \mathcal{E}').$$

Then $p^{c_2(r)}$ annihilates the image of ι according to (i) and $p^{r_{i+1}(\mathcal{E})}$ annihilates the image of δ according to (a_{i+1}) . So we may take as $r_i(\mathcal{E})$ any number greater than the maximum of $r_{i+1}(\mathcal{E})$ and $c_2(r) + r_{i+1}(\mathcal{E}')$ and obtain the statement (a_i) . In particular, (a_1) holds which proves (ii). \square

4.2. Cohomology of coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -modules. We denote by X_j the reduction of X modulo p^{j+1} .

Proposition 4.2.1. Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module on X and $\hat{\mathcal{E}} = \varprojlim_j \mathcal{E}/p^{j+1}\mathcal{E}$ its p -adic completion, which we consider as a sheaf on \mathfrak{X} .

(i) For all $i \geq 0$ one has $H^i(\mathfrak{X}, \hat{\mathcal{E}}) = \varprojlim_j H^i(X_j, \mathcal{E}/p^{j+1}\mathcal{E})$.

(ii) For all $i > 0$ one has $H^i(\mathfrak{X}, \hat{\mathcal{E}}) = H^i(X, \mathcal{E})$.

(iii) $H^0(\mathfrak{X}, \hat{\mathcal{E}}) = \varprojlim_j H^0(X, \mathcal{E})/p^{j+1}H^0(X, \mathcal{E})$.

Proof. Put $\mathcal{E}_j = \mathcal{E}/p^{j+1}\mathcal{E}$. Let \mathcal{E}_t be the subsheaf defined by

$$\mathcal{E}_t(U) = \mathcal{E}(U)_{\text{tor}} ,$$

where the right hand side denotes the group of torsion elements in $\mathcal{E}(U)$. This is indeed a sheaf (and not only a presheaf) because X is a noetherian space. Furthermore, \mathcal{E}_t is a $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -submodule of \mathcal{E} . Because the sheaf $\tilde{\mathcal{D}}_{X,k}^{(m)}$ has noetherian rings of sections over open affine subsets of X , cf. 3.2.9, the submodule \mathcal{E}_t is a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module. \mathcal{E}_t is thus generated by a coherent \mathcal{O}_X -submodule \mathcal{F} of \mathcal{E}_t . The submodule \mathcal{F} is annihilated by a fixed power p^c of p , and so is \mathcal{E}_t . Put $\mathcal{G} = \mathcal{E}/\mathcal{E}_t$, which is again a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module. Using 4.1.9, we can then assume, after possibly replacing c by a larger number, that

- (a) $p^c \mathcal{E}_t = 0$,
- (b) for all $i > 0$: $p^c H^i(X, \mathcal{E}) = 0$,
- (c) for all $i > 0$: $p^c H^i(X, \mathcal{G}) = 0$.

From here on the proof of the proposition is exactly as in [18, 4.2.1]. \square

Proposition 4.2.2. *Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -module.*

(i) *There is $r_1(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geq r_1(\mathcal{E})$ there is $s \in \mathbb{Z}_{\geq 0}$ and an epimorphism of $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -modules*

$$\left(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathcal{E} .$$

(ii) *There is $r_2(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geq r_2(\mathcal{E})$ and all $i > 0$*

$$H^i(\mathfrak{X}, \mathcal{E}(r)) = 0 .$$

Proof. (i) Because \mathcal{E} is a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -module, and because $H^0(U, \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})$ is a noetherian ring for all open affine subsets $U \subset \mathfrak{X}$, cf. 3.3.9, the torsion submodule $\mathcal{E}_t \subset \mathcal{E}$ is again a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -module. As \mathfrak{X} is quasi-compact, there is $c \in \mathbb{Z}_{\geq 0}$ such that $p^c \mathcal{E}_t = 0$. Put $\mathcal{G} = \mathcal{E}/\mathcal{E}_t$ and $\mathcal{G}_0 = \mathcal{G}/p\mathcal{G}$. For $j \geq c$ one has an exact sequence

$$0 \rightarrow \mathcal{G}_0 \xrightarrow{p^{j+1}} \mathcal{E}_{j+1} \rightarrow \mathcal{E}_j \rightarrow 0 .$$

We note that the sheaf \mathcal{G}_0 is a coherent module over $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}/p\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$. We view \mathfrak{X} as a closed subset of X and denote the closed embedding temporarily by i . Because the canonical map of sheaves of rings

$$(4.2.3) \quad \tilde{\mathcal{D}}_{X,k}^{(m)}/p\tilde{\mathcal{D}}_{X,k}^{(m)} \xrightarrow{\simeq} i_* \left(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}/p\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)} \right)$$

is an isomorphism, $i_*\mathcal{G}_0$ can be considered a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module via this isomorphism. Hence we can apply 4.1.8 to $i_*\mathcal{G}_0$ and deduce that there is $r_2(\mathcal{G}_0)$ such that for all $r \geq r_2(\mathcal{G}_0)$ one has

$$H^1(\mathfrak{X}, \mathcal{G}_0(r)) = H^1(X, i_*\mathcal{G}_0(r)) = 0.$$

The canonical maps

$$(4.2.4) \quad H^0(\mathfrak{X}, \mathcal{E}_{j+1}(r)) \longrightarrow H^0(\mathfrak{X}, \mathcal{E}_j(r))$$

are thus surjective for $r \geq r_2(\mathcal{G}_0)$ and $j \geq c$. Similarly, \mathcal{E}_c is a coherent module over $\tilde{\mathcal{D}}_{X,k}^{(m)}/p^c\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module, in particular a coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -module. By 4.1.8 there is $r_1(\mathcal{E}_c)$ such that for every $r \geq r_1(\mathcal{E}_c)$ there is $s \in \mathbb{Z}_{\geq 0}$ and a surjection

$$\lambda : \left(\tilde{\mathcal{D}}_{X,k}^{(m)}/p^c\tilde{\mathcal{D}}_{X,k}^{(m)} \right)^{\oplus s} \twoheadrightarrow \mathcal{E}_c(r).$$

Let $r_1(\mathcal{E}) = \max\{r_2(\mathcal{G}_0), r_1(\mathcal{E}_c)\}$, and assume from now on that $r \geq r_1(\mathcal{E})$. Let e_1, \dots, e_s be the standard basis of the domain of λ , and use 4.2.4 to lift each $\lambda(e_t)$, $1 \leq t \leq s$, to an element of

$$\varprojlim_j H^0(\mathfrak{X}, \mathcal{E}_j(r)) \simeq H^0(\mathfrak{X}, \widehat{\mathcal{E}(r)}),$$

by 4.2.1 (i). But $\widehat{\mathcal{E}(r)} = \widehat{\mathcal{E}}(r)$, and $\widehat{\mathcal{E}} = \mathcal{E}$, as follows from [5, 3.2.3 (v)]. This defines a morphism

$$\left(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)} \right)^{\oplus s} \longrightarrow \mathcal{E}(r)$$

which is surjective because, modulo p^c , it is a surjective morphism of sheaves coming from coherent $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules by reduction modulo p^c , cf. [5, 3.2.2 (ii)].

(ii) We deduce from 4.1.4 and 4.2.1 that for all $i > 0$

$$H^i\left(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}(r)\right) = 0,$$

whenever $r \geq r_0$, where r_0 is as in 3.2.5. Since the sheaf $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ is coherent, cf. 3.3.9, and \mathfrak{X} is a noetherian space of finite dimension, the statement in (ii) can now be deduced by descending induction on i exactly as in the proof of part (ii) of 4.1.8. \square

Proposition 4.2.5. *Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -module.*

(i) *There is $c = c(\mathcal{E}) \in \mathbb{Z}_{\geq 0}$ such that for all $i > 0$ the cohomology group $H^i(\mathfrak{X}, \mathcal{E})$ is annihilated by p^c .*

(ii) $H^0(\mathfrak{X}, \mathcal{E}) = \varprojlim_j H^0(\mathfrak{X}, \mathcal{E})/p^j H^0(\mathfrak{X}, \mathcal{E})$.

Proof. (i) Let $r \in \mathbb{Z}$. By 4.2.1 we have for $i > 0$ that

$$H^i(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}(-r)) = H^i(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(-r)) ,$$

and this is annihilated by a finite power of p , by 4.1.9. The proof now proceeds by descending induction exactly as in the proof of part (ii) of 4.1.9.

(ii) Let $\mathcal{E}_t \subset \mathcal{E}$ be the subsheaf of torsion elements and $\mathcal{G} = \mathcal{E}/\mathcal{E}_t$. Then the discussion in the beginning of the proof of 4.2.1 shows that there is $c \in \mathbb{Z}_{\geq 0}$ such that $p^c \mathcal{E}_t = 0$. Part (i) gives that $p^c H^1(\mathfrak{X}, \mathcal{E}) = p^c H^1(\mathfrak{X}, \mathcal{G}) = 0$, after possibly increasing c . Now we can apply the same reasoning as in the proof of 4.2.1 (iii) to conclude that assertion (ii) is true. \square

4.2.6. Let $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})$ (resp. $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})$) be the category of coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -modules (resp. $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -modules). Let $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})_{\mathbb{Q}}$ be the category of coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -modules up to isogeny. We recall that this means that $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})_{\mathbb{Q}}$ has the same class of objects as $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})$, and for any two objects \mathcal{M} and \mathcal{N} one has

$$\text{Hom}_{\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})_{\mathbb{Q}}}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})}(\mathcal{M}, \mathcal{N}) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Proposition 4.2.7. (i) *The functor $\mathcal{M} \mapsto \mathcal{M}_{\mathbb{Q}} = \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an equivalence between $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)})_{\mathbb{Q}}$ and $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})$.*

(ii) *For every coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module \mathcal{M} there is $m \geq 0$ and a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module \mathcal{M}_m and an isomorphism of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules*

$$\varepsilon : \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{M}_m \xrightarrow{\simeq} \mathcal{M} .$$

If $(m', \mathcal{M}_{m'}, \varepsilon')$ is another such triple, then there is $\ell \geq \max\{m, m'\}$ and an isomorphism of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)}$ -modules

$$\varepsilon_\ell : \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)} \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{M}_m \xrightarrow{\simeq} \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)} \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m')}} \mathcal{M}_{m'}$$

such that $\varepsilon' \circ \left(\text{id}_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger} \otimes \varepsilon_\ell \right) = \varepsilon$.

Proof. (i) This is [5, 3.4.5]. Note that the sheaf $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ satisfies the conditions in [5, 3.4.1], by 3.3.9. We point out that the formal scheme \mathcal{X} in [5, sec. 3.4] is not supposed to be smooth over a discrete valuation ring, but only locally noetherian, cf. [5, sec. 3.3].

(ii) This is [5, 3.6.2]. In this reference the formal scheme is supposed to be noetherian and quasi-separated, but not necessarily smooth over a discrete valuation ring. \square

Theorem 4.2.8. *Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module (resp. $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module).*

(i) *There is $r(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geq r(\mathcal{E})$ there is $s \in \mathbb{Z}_{\geq 0}$ and an epimorphism of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -modules (resp. $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules)*

$$\left(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathcal{E} \quad (\text{ resp. } \left(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger(-r) \right)^{\oplus s} \twoheadrightarrow \mathcal{E}) .$$

(ii) *For all $i > 0$ one has $H^i(\mathfrak{X}, \mathcal{E}) = 0$.*

Proof. (a) We first show both assertions (i) and (ii) for a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module \mathcal{E} . By 4.2.7 (i) there is a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}$ -module \mathcal{F} such that $\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{E}$. We use 4.2.2 to find for every $r \geq r_1(\mathcal{F})$ a surjection

$$\left(\tilde{\mathcal{D}}_{\mathfrak{X},k}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathcal{F} ,$$

for some s (depending on r). Tensoring with \mathbb{Q} gives then the desired surjection onto \mathcal{E} . Hence assertion (i). Furthermore, for $i > 0$

$$H^i(\mathfrak{X}, \mathcal{E}) = H^i(\mathfrak{X}, \mathcal{F}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 ,$$

by 4.2.5, and this proves (ii).

(b) Now suppose \mathcal{E} is a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module. By 4.2.7 (ii) there is $m \geq 0$ and a coherent module \mathcal{E}_m over $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ and an isomorphism of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules

$$\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{E}_m \xrightarrow{\sim} \mathcal{E} .$$

Now use what we have just shown for \mathcal{E}_m in (a) and get the sought for surjection after tensoring with $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. This proves the first assertion. We have

$$\mathcal{E} = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{E}_m = \varinjlim_{\ell \geq m} \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)} \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{E}_m = \varinjlim_{\ell \geq m} \mathcal{E}_\ell$$

where $\mathcal{E}_\ell = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)} \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}} \mathcal{E}_m$ is a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(\ell)}$ -module. Then we have for $i > 0$

$$H^i(\mathfrak{X}, \mathcal{E}) = \varinjlim_{\ell \geq m} H^i(\mathfrak{X}, \mathcal{E}_\ell) = 0 ,$$

by part (a). And this proves assertion (ii). \square

4.3. \mathfrak{X} is $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -affine and $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -affine.

Proposition 4.3.1. (i) Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module. Then \mathcal{E} is generated by its global sections as $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module. Furthermore, \mathcal{E} has a resolution by finite free $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -modules.

(ii) Let \mathcal{E} be a coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module. Then \mathcal{E} is generated by its global sections as $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module. $H^0(\mathfrak{X}, \mathcal{E})$ is a $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger)$ -module of finite presentation. Furthermore, \mathcal{E} has a resolution by finite free $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules.

Proof. (i) Using 4.2.8 it remains to see that any $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -module of type $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}(-r)$ admits a linear surjection $(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})^{\oplus s} \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}(-r)$ for suitable $s \geq 0$. We argue as in [13, 5.1]. Let $M := H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)}(-r))$, a finitely generated $D^{(m)}(\mathbb{G}(k))$ -module by 3.3.6. Consider the linear map of $\tilde{\mathcal{D}}_{X,k}^{(m)}$ -modules equal to the composite

$$\tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{D^{(m)}(\mathbb{G}(k))} M \rightarrow \tilde{\mathcal{D}}_{X,k}^{(m)} \otimes_{H^0(X, \tilde{\mathcal{D}}_{X,k}^{(m)})} M \rightarrow \tilde{\mathcal{D}}_{X,k}^{(m)}(-r)$$

where the first map is the surjection induced by the map $Q_{X,k}^{(m)}$ appearing in 3.3.3. Let \mathcal{E} be the cokernel of the composite map. Since $D^{(m)}(\mathbb{G}(k))$ is noetherian, the source of the map is coherent and hence \mathcal{E} is coherent. Moreover, $\mathcal{E} \otimes \mathbb{Q} = 0$ since $\tilde{\mathcal{D}}_{X,k}^{(m)}(-r) \otimes \mathbb{Q}$ is generated by global sections [4]. All in all, there is i with $p^i \mathcal{E} = 0$. Now choose a linear surjection $(D^{(m)}(\mathbb{G}(k)))^{\oplus s} \rightarrow M$. We obtain the exact sequence of coherent modules

$$(\tilde{\mathcal{D}}_{X,k}^{(m)})^{\oplus s} \rightarrow \tilde{\mathcal{D}}_{X,k}^{(m)}(-r) \rightarrow \mathcal{E} \rightarrow 0.$$

Passing to p -adic completions (which is exact in our situation [5, 3.2]) and inverting p yields the linear surjection

$$(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})^{\oplus s} \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}(-r).$$

This shows (i).

(ii) This follows from (i) exactly as in [13]. \square

4.3.2. The functors $\mathcal{L}oc_{\mathfrak{X},k}^{(m)}$ and $\mathcal{L}oc_{\mathfrak{X},k}^\dagger$. Let E be a finitely generated $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})$ -module (resp. a finitely presented $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger)$ -module). Then we let $\mathcal{L}oc_{\mathfrak{X},k}^{(m)}(E)$ (resp. $\mathcal{L}oc_{\mathfrak{X},k}^\dagger(E)$) be the sheaf on \mathfrak{X} associated to the presheaf

$$U \mapsto \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}(U) \otimes_{H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})} E \quad (\text{resp. } U \mapsto \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger}(U) \otimes_{H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger})} E).$$

It is obvious that $\mathcal{L}oc_{\mathfrak{X},k}^{(m)}$ (resp. $\mathcal{L}oc_{\mathfrak{X},k}^{\dagger}$) is a functor from the category of finitely generated $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})$ -modules (resp. finitely presented $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger})$ -modules) to the category of sheaves of modules over $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ (resp. $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger}$).

Theorem 4.3.3. (i) *The functors $\mathcal{L}oc_{\mathfrak{X},k}^{(m)}$ and H^0 (resp. $\mathcal{L}oc_{\mathfrak{X},k}^{\dagger}$ and H^0) are quasi-inverse equivalences between the categories of finitely generated $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)})$ -modules and coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{(m)}$ -modules (resp. finitely presented $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger})$ -modules and coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger}$ -modules).*

(ii) *The functor $\mathcal{L}oc_{\mathfrak{X},k}^{(m)}$ (resp. $\mathcal{L}oc_{\mathfrak{X},k}^{\dagger}$) is an exact functor.*

Proof. The proofs of [13, 5.2.1, 5.2.3] for the first and the second assertion, respectively, carry over word for word. \square

5. LOCALIZATION OF REPRESENTATIONS OF $\mathbb{G}(L)$

5.1. Modules over $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\theta_0}$. Let as before \mathfrak{X} be the p -adic completion of an admissible blow up X of X_0 . We recall that the algebra $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^{\dagger})$ is canonically isomorphic to the coherent L -algebra $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\theta_0}$, cf. 3.3.9. According to [18, Lem. 5.2.1] the algebras $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})$ and $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\theta_0}$ are compact type algebras with noetherian defining Banach algebras in the sense of loc.cit.

In the following we extend some notions appearing in [18, 5.2] to the more general situation considered here. We consider the locally L -analytic compact group $G_0 = \mathbb{G}_0(\mathfrak{o})$ with its series of congruence subgroups $G_{k+1} = \mathbb{G}(k)^{\circ}(L)$. The group G_0 acts by translations on the space $C^{\text{cts}}(G_0, K)$ of continuous K -valued functions. Following [10, (5.3)] let $D(\mathbb{G}(k)^{\circ}, G_0)$ be the strong dual of the space of $\mathbb{G}(n)^{\circ}$ -analytic vectors

$$D(\mathbb{G}(k)^{\circ}, G_0) := (C^{\text{cts}}(G_0, K)_{\mathbb{G}(k)^{\circ}\text{-an}})'_b.$$

It is a locally convex topological L -algebra naturally isomorphic to the crossed product of the ring $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})$ with the finite group G_0/G_{k+1} . In particular,

$$(5.1.1) \quad D(\mathbb{G}(k)^{\circ}, G_0) = \bigoplus_{g \in G_0/G_{k+1}} \mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ}) * \delta_g$$

is a finitely generated free topological module over $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})$. Denoting by $C^{\text{la}}(G_0, K)$ the space of K -valued locally analytic functions and dualizing the isomorphism

$$\varinjlim_k C^{\text{cts}}(G_0, K)_{\mathbb{G}(k)^\circ - \text{an}} \xrightarrow{\simeq} C^{\text{la}}(G_0, K)$$

yields an isomorphism of topological algebras

$$D(G_0) \xrightarrow{\simeq} \varprojlim_k D(\mathbb{G}(k)^\circ, G_0) .$$

This is the weak Fréchet-Stein structure on the locally analytic distribution algebra $D(G_0)$ as introduced by Emerton in [10, Prop. 5.3.1]. In an obviously similar manner, we may construct the ring $D(\mathbb{G}(k)^\circ, G_0)_{\theta_0}$ and obtain an isomorphism $D(G_0)_{\theta_0} \xrightarrow{\simeq} \varprojlim_k D(\mathbb{G}(k)^\circ, G_0)_{\theta_0}$.

We consider an admissible locally analytic G_0 -representation V , its coadmissible module $M := V'_b$ and its subspace of $\mathbb{G}(k)^\circ$ -analytic vectors $V_{\mathbb{G}(k)^\circ - \text{an}} \subseteq V$. The latter subspace is naturally a nuclear Fréchet space [10, Lem. 6.1.6] and we let $(V_{\mathbb{G}(k)^\circ - \text{an}})'_b$ be its strong dual. It is a space of compact type and a topological $D(\mathbb{G}(k)^\circ, G_0)$ -module which is finitely generated [10, Lem. 6.1.13]. According to [10, Thm. 6.1.20] the modules $M_k := (V_{\mathbb{G}(k)^\circ - \text{an}})'$ form a $(D(\mathbb{G}(k)^\circ, G_0))_{k \in \mathbb{N}}$ -sequence, in the sense of [10, Def. 1.3.8], for the coadmissible module M relative to the weak Fréchet-Stein structure on $D(G_0)$. This implies that one has

$$(5.1.2) \quad M_k = D(\mathbb{G}(k)^\circ, G_0) \hat{\otimes}_{D(G_0)} M$$

as $D(\mathbb{G}(k)^\circ, G_0)$ -modules for any k . Here, the completed tensor product is understood in the sense of [10, Lem. 1.2.3], as in [18].

Lemma 5.1.3. (i) *The $D(\mathbb{G}(k)^\circ, G_0)$ -module M_k is finitely presented.*

(ii) *There are natural isomorphisms*

$$D(\mathbb{G}(k-1)^\circ, G_0) \otimes_{D(\mathbb{G}(k)^\circ, G_0)} M_k \xrightarrow{\simeq} M_{k-1} .$$

Proof. This can be proved exactly as [18, Lem. 5.2.4]. □

Remark: These results have obvious analogues when the character θ_0 is involved.

5.2. G_0 -equivariance and the functor $\mathcal{L}oc_\infty^\dagger$. A p -adic completion \mathfrak{X} of an admissible blow-up X of X_0 will be called an *admissible formal blow-up* of \mathfrak{X}_0 . We note here that any formal scheme \mathfrak{X} which is obtained from \mathfrak{X}_0 by blowing-up a coherent open ideal on \mathfrak{X}_0 is an admissible blow-up in this sense. Indeed, if $\mathcal{I} \subset \mathfrak{X}_0$ is the ideal which is blown-up, then $\mathcal{I} \cap \mathcal{O}_{X_0}$ is a coherent open ideal on X_0 . Blowing-up this ideal on X_0 and completing p -adically gives back \mathfrak{X} .

If the coherent open ideal $\mathcal{I} \subset X_0$ that is blown-up is G_0 -stable, then there is an induced G_0 -action on X and \mathfrak{X} . In this case, we will say that X and \mathfrak{X} are G_0 -stable. In the following we suppose that k is large enough for \mathfrak{X} , so that the sheaf $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ is defined on \mathfrak{X} .

Proposition 5.2.1. *Let $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a morphism between admissible formal blow-ups of \mathfrak{X}_0 which is an isomorphism on corresponding rigid analytic spaces. If \mathcal{M}' is a coherent $\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger$ -module, then $R^j\pi_*\mathcal{M}' = 0$ for $j > 0$. Moreover, $\pi_*\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$, so that π_* induces an exact functor between coherent modules over $\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger$ and $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ respectively.*

Proof. We denote the associated rigid analytic space of \mathfrak{X} by $\mathfrak{X}_{\mathbb{Q}}$. Coherent modules over $\mathfrak{X}_{\mathbb{Q}}$ are equivalent to coherent modules over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ [5, 4.1.3] and similarly for \mathfrak{X}' . This implies $R^j\pi_*\mathcal{O}_{\mathfrak{X}',\mathbb{Q}} = 0$ for $j > 0$ and $\pi_*\mathcal{O}_{\mathfrak{X}',\mathbb{Q}} = \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. In particular, there is $N \geq 0$ such that $p^N R^j\pi_*\mathcal{O}_{\mathfrak{X}'} = 0$ for $j > 0$ and such that the kernel and cokernel of the natural map $\mathcal{O}_{\mathfrak{X}} \rightarrow \pi_*\mathcal{O}_{\mathfrak{X}'}$ are killed by p^N . For any $i \geq 0$, let X_i be the reduction of $\mathfrak{X} \bmod p^{i+1}$ and similarly for \mathfrak{X}' and denote by $\pi_i : X'_i \rightarrow X_i$ the morphism induced by π . For any i we have then $p^N R^j\pi_{i*}\mathcal{O}_{X'_i} = 0$ for $j > 0$ and such that the kernel and cokernel of the natural map $\mathcal{O}_{X_i} \rightarrow \pi_{i*}\mathcal{O}_{X'_i}$ are killed by p^N . We now prove the assertions. According to 4.3.1 the module \mathcal{M}' is generated by global sections. Induction on j reduces us therefore to the case $\mathcal{M}' = \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger$. Since $R^j\pi_*$ commutes with inductive limits, it suffices to prove the claim for $\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^{(m)}$. Abbreviate $\tilde{\mathcal{D}}_{X_i,k'}^{(m)}$ for $\tilde{\mathcal{D}}_{\mathfrak{X},k'}^{(m)}/p^{i+1}\tilde{\mathcal{D}}_{\mathfrak{X},k'}^{(m)}$ and similarly for \mathfrak{X}' . Then $\tilde{\mathcal{D}}_{X'_i,k'}^{(m)} = \pi_i^*\tilde{\mathcal{D}}_{X_i,k'}^{(m)}$ by 3.2.9 and the projection formula implies

$$R^j\pi_{i*}\tilde{\mathcal{D}}_{X'_i,k'}^{(m)} = \tilde{\mathcal{D}}_{X_i,k'}^{(m)} \otimes R^j\pi_{i*}\mathcal{O}_{X'_i}.$$

We see for any $i \geq 0$ that

$$p^N R^j\pi_{i*}\tilde{\mathcal{D}}_{X'_i,k'}^{(m)} = 0$$

for all $j > 0$ and that the natural map

$$\tilde{\mathcal{D}}_{X_i,k'}^{(m)} \rightarrow \pi_{i*}\tilde{\mathcal{D}}_{X'_i,k'}^{(m)}$$

has kernel and cokernel killed by p^N . Taking inverse limits over i , arguing as in 4.2.1 and finally inverting p yields the first claim and $\pi_*\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. \square

5.2.2. If \mathfrak{X} is G_0 -stable, then there is an induced (left) action of G_0 on the sheaf $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. Given $g \in G_0$ and a local section s of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$, there is thus a local section $g.s$ of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$. We can then consider the abelian category $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger, G_0)$ of (left) G_0 -equivariant coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules. Furthermore, the group G_{k+1} is contained in $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)$ as a set of delta distributions, and for $h \in G_{k+1}$ we write δ_h for its image in $H^0(\mathfrak{X}, \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger) = \mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$. For $g \in G_0$, $h \in G_{k+1}$, we have $g.\delta_h = \delta_{ghg^{-1}}$, and for a local section s of $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ we have then the relation

$$(5.2.3) \quad g.(s\delta_h) = (g.s)(g.\delta_h) = (g.s)\delta_{ghg^{-1}} .$$

Suppose now that $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}$ is a G_0 -equivariant morphism between admissible formal blow-ups of \mathfrak{X}_0 which is an isomorphism on corresponding rigid analytic spaces and that $k' \geq k$ are sufficiently large. According to 5.2.1 there is then a natural morphism of sheaves of rings

$$(5.2.4) \quad \pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \xrightarrow{\subseteq} \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$$

which is G_0 -equivariant. Given a coherent $\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger$ -module $\mathcal{M}_{\mathfrak{X}'}$, the $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module

$$\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger} \pi_* \mathcal{M}_{\mathfrak{X}'}$$

is G_0 -equivariant via $g.(s \otimes m) = (g.s) \otimes (g.m)$ for local sections s, m and $g \in G_0$. Consider its submodule $\mathcal{R}_{\mathfrak{X}}$ locally generated by all elements $s\delta_h \otimes m - s \otimes (h.m)$ for $h \in G_{k'}$. Because of 5.2.3 the submodule $\mathcal{R}_{\mathfrak{X}}$ is G_0 -stable. We put

$$\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger, G_{k'}} \mathcal{M}_{\mathfrak{X}'} := \left(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger} \pi_* \mathcal{M}_{\mathfrak{X}'} \right) / \mathcal{R}_{\mathfrak{X}} .$$

5.2.5. We let $X = \mathbb{G}/\mathbb{B}$ be the flag variety of \mathbb{G} and denote by

$$\mathcal{X} := X^{\text{ad}}$$

the associated adic space. For simplicity, an admissible formal blow-up \mathfrak{X} of \mathfrak{X}_0 will be called a *formal model for \mathcal{X} over \mathfrak{X}_0* . This set of formal models is a projective system if it is indexed by the directed family of coherent open ideals on \mathfrak{X}_0 , cf. [6, 9.3]. Any morphism in the projective system is an isomorphism on corresponding rigid analytic spaces. Given a subsystem \mathcal{F} of this projective system, we will denote the corresponding projective limit by $\mathfrak{X}_\infty(\mathcal{F}) = \varprojlim_{\mathcal{F}} \mathfrak{X}$ or simply \mathfrak{X}_∞ . In the following we will work relative to such a fixed system \mathcal{F} . A *family of congruence levels for \mathcal{F}* is an increasing family \underline{k} of natural numbers $k = k_{\mathfrak{X}}$ for each $\mathfrak{X} \in \mathcal{F}$ (increasing means that $k' \geq k$ whenever there is a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ in \mathcal{F}). In the following, we fix such a family \underline{k} with the additional property that the sheaf $\tilde{\mathcal{D}}_{\mathfrak{X},k_{\mathfrak{X}},\mathbb{Q}}^\dagger$ is defined for each $\mathfrak{X} \in \mathcal{F}$. We finally assume that all models in \mathcal{F} are G_0 -stable and that all morphisms in \mathcal{F} are G_0 -equivariant.

Proposition 5.2.6. *If \mathcal{F} equals the set of all G_0 -equivariant formal models of \mathcal{X} over \mathfrak{X}_0 , then $\mathfrak{X}_\infty = \mathcal{X}$.*

Proof. According to [19] it suffices to see that any admissible formal blow-up \mathfrak{X} of \mathfrak{X}_0 is dominated by one which is G_0 -stable. If \mathcal{I} is the ideal which is blown-up and if $p^k \mathcal{O}_{\mathfrak{X}_0} \subset \mathcal{I}$ for some k , then $G_k := \mathbb{G}(k)(\mathfrak{o})$ stabilizes \mathcal{I} and $\mathcal{I}' := \mathcal{I} \cdot \mathcal{O}_{\mathfrak{X}}$. Let g_1, \dots, g_N be a system

of representatives for G_0/G_k and let \mathcal{J} be the product of the finitely many ideals $g_i \mathcal{I}'$. Then \mathcal{J} is G_0 -stable and blowing-up \mathcal{J} on \mathfrak{X} yields a G_0 -stable model over \mathfrak{X} . \square

Definition 5.2.7. A G_0 -equivariant coadmissible module on \mathfrak{X}_∞ consists of a family $\mathcal{M} := (\mathcal{M}_{\mathfrak{X}})_{\mathfrak{X} \in \mathcal{F}}$ of objects $\mathcal{M}_{\mathfrak{X}} \in \text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger, G_0)$ together with isomorphisms

$$(5.2.8) \quad \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger, G_{k'}} \mathcal{M}_{\mathfrak{X}'} \xrightarrow{\simeq} \mathcal{M}_{\mathfrak{X}}$$

of G_0 -equivariant $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -modules whenever there is a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ in \mathcal{F} . The isomorphisms are required to satisfy the obvious transitivity condition whenever there are morphisms $\mathfrak{X}'' \rightarrow \mathfrak{X}' \rightarrow \mathfrak{X}$ in \mathcal{F} .

A morphism $\mathcal{M} \rightarrow \mathcal{N}$ between two such modules consists of morphisms $\mathcal{M}_{\mathfrak{X}} \rightarrow \mathcal{N}_{\mathfrak{X}}$ in $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger, G_0)$ compatible with the isomorphisms above.

Let \mathcal{M} be a G_0 -equivariant coadmissible module on \mathfrak{X}_∞ . The isomorphisms 5.2.8 induce morphisms $\pi_* \mathcal{M}_{\mathfrak{X}'} \rightarrow \mathcal{M}_{\mathfrak{X}}$ having global sections $H^0(\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'} \rightarrow H^0(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We let

$$H^0(\mathfrak{X}_\infty, \mathcal{M}) := \varprojlim_{\mathfrak{X}} H^0(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}).$$

5.2.9. On the other hand, we consider the category of coadmissible $D(G_0)_{\theta_0}$ -modules. Given such a module M we have its associated admissible locally analytic G_0 -representation $V = M'_b$ together with its subspace of $\mathbb{G}(k)^\circ$ -analytic vectors $V_{\mathbb{G}(k)^\circ\text{-an}}$. The latter is stable under the G_0 -action and its dual $M_k := (V_{\mathbb{G}(k)^\circ\text{-an}})'$ is a finitely presented $D(\mathbb{G}(k)^\circ, G_0)_{\theta_0}$ -module, cf. 5.1.3. Now consider a model \mathfrak{X} in \mathcal{F} and let $k = k_{\mathfrak{X}}$. According to Thm. 4.3.3 we have the coherent $\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ -module

$$\mathcal{L}oc_{\mathfrak{X},k}^\dagger(M_k) = \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_{\theta_0}} M_k$$

on \mathfrak{X} . Using the contragredient G_0 -action on the dual space M_k , we put

$$g.(s \otimes m) := (g.s) \otimes (g.m)$$

for $g \in G_0, m \in M_k$ and a local section s . In this way, $\mathcal{L}oc_{\mathfrak{X},k}^\dagger(M_k)$ becomes an object of $\text{Coh}(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger, G_0)$.

Proposition 5.2.10. (i) The family $\mathcal{L}oc_{\mathfrak{X},k_{\mathfrak{X}}}^\dagger(M_{k_{\mathfrak{X}}})$ forms a G_0 -equivariant coadmissible module on \mathfrak{X}_∞ . Call it $\mathcal{L}oc^\dagger(M)$. The formation of $\mathcal{L}oc^\dagger(M)$ is functorial in M .

(ii) The functors $\mathcal{L}oc^\dagger$ and $H^0(\mathfrak{X}_\infty, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G_0)_{\theta_0}$ -modules and G_0 -equivariant coadmissible modules on \mathfrak{X}_∞ .

Proof. Assume $k' \geq k$. We let $H := G_{k+1}/G_{k'+1}$ and we denote a system of representatives in G_{k+1} for the cosets in H by the same symbol. For simplicity, we abbreviate in this proof

$$D(k) := \mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_{\theta_0} \quad \text{and} \quad D(k, G_0) := D(\mathbb{G}(k)^\circ, G_0)_{\theta_0}$$

and similarly for k' . We have the natural inclusion $D(k) \hookrightarrow D(k, G_0)$ from 5.1.1 which is compatible with variation in k . Now suppose M is a $D(k', G_0)$ -module. We then have the free $D(k)$ -module $D(k)^{\oplus M \times H}$ on a basis $e_{m,h}$ indexed by the elements (m, h) of the set $M \times H$. Its formation is functorial in M : if M' is another module and $f : M \rightarrow M'$ a linear map, then $e_{m,h} \rightarrow e_{f(m),h}$ induces a linear map between the corresponding free modules. The free module comes with a linear map

$$f_M : D(k)^{\oplus M \times H} \rightarrow D(k) \otimes_{D(k')} M$$

given by

$$\bigoplus_{(m,h)} \lambda_{m,h} e_{m,h} \mapsto (\lambda_{m,h} \delta_h) \otimes m - \lambda_{m,h} \otimes (h \cdot m)$$

for $\lambda_{m,h} \in D(k)$ where we consider M a $D(k')$ -module via the inclusion $D(k') \hookrightarrow D(k', G_0)$. The map is visibly functorial in M and gives rise to the sequence of linear maps

$$D(k)^{\oplus M \times H} \xrightarrow{f_M} D(k) \otimes_{D(k')} M \xrightarrow{\text{can}_M} D(k, G_0) \otimes_{D(k', G_0)} M \longrightarrow 0$$

where the second map is induced from the inclusion $D(k') \hookrightarrow D(k', G_0)$. The sequence is functorial in M , since so are both occurring maps.

Claim 1: If M is a finitely presented $D(k', G_0)$ -module, then the above sequence is exact.

Proof. This can be proved as in the proof of [18, Prop. 5.3.5]. \square

Let $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a morphism in \mathcal{F} and let $k = k_{\mathfrak{X}}, k' = k_{\mathfrak{X}'}$.

Claim 2: Suppose M is a finitely presented $D(k')$ -module and let $\mathcal{M} := \mathcal{L}oc_{\mathfrak{X}', k'}^\dagger(M)$. The natural morphism

$$\mathcal{L}oc_{\mathfrak{X}, k}^\dagger(D(k) \otimes_{D(k')} M) \xrightarrow{\cong} \tilde{\mathcal{D}}_{\mathfrak{X}, k, \mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}', k', \mathbb{Q}}^\dagger} \pi_* \mathcal{M}$$

is bijective.

Proof. The functor π_* is exact on coherent $\tilde{\mathcal{D}}_{\mathfrak{X}', k', \mathbb{Q}}^\dagger$ -modules according to 5.2.1. Choosing a finite presentation of M reduces to the case $M = D(k')$ which is obvious. \square

Now let M be a finitely presented $D(k', G_0)$ -module. Let m_1, \dots, m_r be generators for M as a $D(k')$ -module. We have a sequence of $D(k)$ -modules

$$\bigoplus_{i,h} D(k)e_{m_i,h} \xrightarrow{f'_M} D(k) \otimes_{D(k')} M \xrightarrow{\text{can}_M} D(k, G_0) \otimes_{D(k', G_0)} M \longrightarrow 0$$

where f'_M denotes the restriction of the map f_M to the free submodule of $D(k)^{\oplus M \times H}$ generated by the finitely many vectors $e_{m_i,h}, i = 1, \dots, r, h \in H_n$. Since $\text{im}(f'_M) = \text{im}(f_M)$ the sequence is exact by the first claim. Since it consists of finitely presented $D(k)$ -modules, we may apply the exact functor $\mathcal{L}oc_{\mathfrak{X},k,\mathbb{Q}}^\dagger$ to it. By the second claim, we get an exact sequence

$$(\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger)^{\oplus r|H|} \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger} \pi_* \mathcal{M} \rightarrow \mathcal{L}oc_{\mathfrak{X},k}^\dagger(D(k, G_0) \otimes_{D(k', G_0)} M) \rightarrow 0$$

where $\mathcal{M} = \mathcal{L}oc_{\mathfrak{X}',k'}^\dagger(M)$. The cokernel of the first map in this sequence equals by definition

$$\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger, G_{k'}} \pi_* \mathcal{M} ,$$

whence an isomorphism

$$\tilde{\mathcal{D}}_{\mathfrak{X},k,\mathbb{Q}}^\dagger \otimes_{\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger, G_k'} \pi_* \mathcal{M} \xrightarrow{\cong} \mathcal{L}oc_{\mathfrak{X},k}^\dagger(D(k, G_0) \otimes_{D(k', G_0)} M) .$$

This implies both parts of the proposition. \square

5.2.11. Denote the canonical projection map $\mathfrak{X}_\infty \rightarrow \mathfrak{X}$ by $\text{sp}_\mathfrak{X}$ for each \mathfrak{X} . We define the following sheaf of rings on \mathfrak{X}_∞ . Assume $V \subseteq \mathfrak{X}_\infty$ is an open subset of the form $\text{sp}_\mathfrak{X}^{-1}(U)$ with an open subset $U \subseteq \mathfrak{X}$ for a model \mathfrak{X} . We have that

$$\text{sp}_{\mathfrak{X}'}(V) = \pi^{-1}(U)$$

for any blow-up morphism $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}$ in \mathcal{F} and so, in particular, $\text{sp}_{\mathfrak{X}'}(V) \subseteq \mathfrak{X}'$ is an open subset for such \mathfrak{X}' . Moreover,

$$\pi^{-1}(\text{sp}_{\mathfrak{X}'}(V)) = \text{sp}_{\mathfrak{X}''}(V)$$

whenever $\pi : \mathfrak{X}'' \rightarrow \mathfrak{X}'$ is a blow-up morphism over \mathfrak{X} in \mathcal{F} . In this situation, the morphism 5.2.4 induces the ring homomorphism

$$(5.2.12) \quad \tilde{\mathcal{D}}_{\mathfrak{X}'',k'',\mathbb{Q}}^\dagger(\text{sp}_{\mathfrak{X}''}(V)) = \pi_* \tilde{\mathcal{D}}_{\mathfrak{X}'',k'',\mathbb{Q}}^\dagger(\text{sp}_{\mathfrak{X}'}(V)) \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X}',k',\mathbb{Q}}^\dagger(\text{sp}_{\mathfrak{X}'}(V))$$

and we form the projective limit

$$\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger(V) := \varprojlim_{\mathfrak{X}' \rightarrow \mathfrak{X}} \tilde{\mathcal{D}}_{\mathfrak{X}', k', \mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}'}(V))$$

over all these maps. The open subsets of the form V form a basis for the topology on \mathfrak{X}_∞ and $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ is a presheaf on this basis. We denote the associated sheaf on \mathfrak{X}_∞ by the symbol $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ as well. It is a G_0 -equivariant sheaf of rings on \mathfrak{X}_∞ .

Remark: In the case where \mathcal{F} consists of *all* G_0 -stable formal models of \mathcal{X} over \mathfrak{X}_0 , we have $\mathfrak{X}_\infty = \mathcal{X}$ by 5.2.6 and we denote the sheaf $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ by $\tilde{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^\dagger$ (or simply $\tilde{\mathcal{D}}_{\mathcal{X}}^\dagger$).

5.2.13. Suppose $\mathcal{M} := (\mathcal{M}_{\mathfrak{X}})_{\mathfrak{X}}$ is a G_0 -equivariant coadmissible module on \mathfrak{X}_∞ as defined in 5.2.7. The isomorphisms 5.2.8 induce G_0 -equivariant maps $\pi_* \mathcal{M}_{\mathfrak{X}'} \rightarrow \mathcal{M}_{\mathfrak{X}}$ which are linear relative to the morphism 5.2.4. In a completely analogous manner as above, we obtain a sheaf \mathcal{M}_∞ on \mathfrak{X}_∞ . It is a G_0 -equivariant (left) $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ -module on \mathfrak{X}_∞ whose formation is functorial in \mathcal{M} .

Proposition 5.2.14. *The functor $\mathcal{M} \rightarrow \mathcal{M}_\infty$ from G_0 -equivariant coadmissible modules on \mathfrak{X}_∞ to G_0 -equivariant $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ -modules is a fully faithful embedding.*

Proof. We have $\mathrm{sp}_{\mathfrak{X}}(\mathfrak{X}_\infty) = \mathfrak{X}$ for all \mathfrak{X} . The global sections of \mathcal{M}_∞ are therefore equal to

$$\Gamma(\mathfrak{X}_\infty, \mathcal{M}_\infty) = \varprojlim_{\mathfrak{X}} \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = H^0(\mathfrak{X}_\infty, \mathcal{M})$$

in the notation of the previous section. Thus, the functor $\mathcal{L}oc^\dagger \circ \Gamma(\mathfrak{X}_\infty, -)$ is a left quasi-inverse according to Prop. 5.2.10. \square

We denote by $\mathcal{L}oc_\infty^\dagger$ the composite of the functor $\mathcal{L}oc^\dagger$ with $(\cdot)_\infty$, i.e.

$$\{ \text{coadmissible } D(G_0)_{\theta_0} - \text{modules} \} \xrightarrow{\mathcal{L}oc_\infty^\dagger} \{ G_0 - \text{equivariant } \tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger - \text{modules} \}.$$

It is fully faithful. We tentatively call its essential image the *coadmissible* G_0 -equivariant $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ -modules. It is an abelian category. In the case where \mathfrak{X}_∞ equals the whole adic flag variety \mathcal{X} we write $\mathcal{L}oc_{\mathcal{X}}^\dagger$ for the functor $\mathcal{L}oc_\infty^\dagger$.

5.3. G -equivariance and the main theorem. Let $G := \mathbb{G}(L)$. Denote by \mathcal{B} the (semi-simple) Bruhat-Tits building of the p -adic group G together with its natural G -action.

5.3.1. To each special vertex $v \in \mathcal{B}$ we have the associated smooth affine Bruhat-Tits group scheme \mathbb{G}_v over \mathfrak{o} and a smooth model $X_0(v)$ of the flag variety of \mathbb{G} . All constructions in sections 3-4 are associated with the group scheme \mathbb{G}_0 with vertex, say v_0 , but can be done canonically for any other of the reductive group schemes \mathbb{G}_v . We distinguish the various constructions from each other by adding the corresponding vertex v to them, i.e. we write $X(v)$ for an admissible blow-up of the smooth model $X_0(v)$ and so on. We

then choose a subset $\mathcal{F}(v)$ of models over $\mathfrak{X}_0(v)$ for \mathcal{X} as in the preceding subsection, but relative to the vertex v . We assume that one of these subsets, say the one belonging to v_0 , is stable under admissible blowing-up. According to [6], any $\mathfrak{X}(v) \in \mathcal{F}(v)$, $v \in \mathcal{B}$ is then dominated by an element from $\mathcal{F}(v_0)$. As before, we denote the projective limit over $\mathcal{F}(v_0)$ by \mathfrak{X}_∞ . According to the previous subsection, we have the G_0 -equivariant sheaf $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ on \mathfrak{X}_∞ .

An element $g \in G$ induces a morphism $X_0(v) \xrightarrow{g\cdot} X_0(gv)$ which satisfies $(gh)\cdot = (g\cdot) \circ (h\cdot)$ and $1\cdot = \text{id}$ for $g, h \in G$. If $X(v) \rightarrow X_0(v)$ is an admissible blowing up of an ideal $\mathcal{I} \subset X_0(v)$, then the universal property of blowing-up induces an isomorphism $X(v) \xrightarrow{g\cdot} X(gv)$ onto the blowing-up of $X_0(gv)$ at the ideal $g\mathcal{I}$. We make the assumption that our union of models is G -stable in the sense that

$$\mathfrak{X}(v) \in \mathcal{F}(v) \implies \mathfrak{X}(gv) \in \mathcal{F}(gv)$$

for any $\mathfrak{X}(v) \in \mathcal{F}(v)$ and any $g \in G$. We also assume that $k_{\mathfrak{X}(v)} = k_{\mathfrak{X}(gv)}$ in this situation. We obtain thus a G -action on \mathfrak{X}_∞ . By definition of this action, there is an equality

$$(5.3.2) \quad \text{sp}_{\mathfrak{X}(gv)}(g.V) = g.\text{sp}_{\mathfrak{X}(v)}(V)$$

in $\mathfrak{X}(gv)$ for $g \in G$ and $V \subseteq \mathfrak{X}_\infty$.

Proposition 5.3.3. *The G_0 -equivariant structure on the sheaf $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ extends to a G -equivariant structure.*

Proof. Let $g \in G$. The isomorphism $\mathfrak{X}(v) \xrightarrow{g\cdot} \mathfrak{X}(gv)$ induces a ring isomorphism

$$(5.3.4) \quad \tilde{\mathcal{D}}_{\mathfrak{X}(v), k, \mathbb{Q}}^\dagger(U) \xrightarrow{g\cdot} \tilde{\mathcal{D}}_{\mathfrak{X}(gv), k, \mathbb{Q}}^\dagger(g.U)$$

for any open subset $U \subseteq \mathfrak{X}(v)$ where $k = k_{\mathfrak{X}(v)}$. In particular, for an open subset $V \subseteq \mathfrak{X}_\infty$ of the form $V = \text{sp}_{\mathfrak{X}(v)}^{-1}(U)$ with $U \subseteq \mathfrak{X}(v)$ open and a blow-up morphism $\mathfrak{X}'(v) \rightarrow \mathfrak{X}(v)$, this gives a ring homomorphism

$$(5.3.5) \quad \tilde{\mathcal{D}}_{\mathfrak{X}'(v), k', \mathbb{Q}}^\dagger(\text{sp}_{\mathfrak{X}'(v)}(V)) \xrightarrow{g\cdot} \tilde{\mathcal{D}}_{\mathfrak{X}'(gv), k', \mathbb{Q}}^\dagger(g.\text{sp}_{\mathfrak{X}'(v)}(V)) = \tilde{\mathcal{D}}_{\mathfrak{X}'(gv), k', \mathbb{Q}}^\dagger(\text{sp}_{\mathfrak{X}'(gv)}(gV))$$

where we have used 5.3.2 and where $k' = k_{\mathfrak{X}'(v)}$. A given morphism $\pi : \mathfrak{X}(v') \rightarrow \mathfrak{X}(v)$ with $\mathfrak{X}(v') \in \mathcal{F}(v')$ and $\mathfrak{X}(v) \in \mathcal{F}(v)$ which is an isomorphism on corresponding rigid analytic spaces induces a morphism of sheaves of rings

$$(5.3.6) \quad \pi_* \tilde{\mathcal{D}}_{\mathfrak{X}(v'), k', \mathbb{Q}}^\dagger = \tilde{\mathcal{D}}_{\mathfrak{X}(v), k', \mathbb{Q}}^\dagger \xrightarrow{\subseteq} \tilde{\mathcal{D}}_{\mathfrak{X}(v), k, \mathbb{Q}}^\dagger$$

by 5.2.1. Here, $k = k_{\mathfrak{X}(v)}$ and $k' = k_{\mathfrak{X}(v')}$. Given $V \subset \mathfrak{X}_\infty$ of the form $V = \mathrm{sp}_{\mathfrak{X}(\tilde{v})}^{-1}(U)$ with an open set $U \subseteq \mathfrak{X}(\tilde{v})$, the morphism 5.3.6 induces a ring homomorphism

$$(5.3.7) \quad \tilde{\mathcal{D}}_{\mathfrak{X}(v'),k',\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v')}(V)) = \pi_* \tilde{\mathcal{D}}_{\mathfrak{X}(v'),k',\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v)}(V)) \rightarrow \tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v)}(V))$$

whenever the morphism $\pi : \mathfrak{X}(v') \rightarrow \mathfrak{X}(v)$ lies over $\mathfrak{X}(\tilde{v})$. If we write $\mathfrak{X}(v') \geq \mathfrak{X}(v)$ in this situation, then the family of all models $\mathfrak{X}(v)$ over $\mathfrak{X}(\tilde{v})$ becomes directed, the $\tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v)}(V))$ become a projective system and we may form the projective limit

$$\varprojlim_{\mathfrak{X}(v) \rightarrow \mathfrak{X}(\tilde{v})} \tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v)}(V)) .$$

By cofinality, this projective limit equals $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger(V)$. Since the homomorphism 5.3.5 is compatible with varying $\mathfrak{X}'(v')$ in the directed family, we deduce for a given $g \in G$ a ring homomorphism

$$\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger(V) = \varprojlim_{\mathfrak{X}(v)} \tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(v)}(V)) \xrightarrow{g} \varprojlim_{\mathfrak{X}(gv)} \tilde{\mathcal{D}}_{\mathfrak{X}(gv),k,\mathbb{Q}}^\dagger(\mathrm{sp}_{\mathfrak{X}(gv)}(gV)) = \tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger(gV) .$$

It implies that the sheaf $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ is G -equivariant. It is clear from the construction that the G -equivariant structure extends the G_0 -structure. \square

A coadmissible G_0 -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -module whose equivariant structure extends to the full group G , will simply be called a coadmissible G -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -module.

Theorem 5.3.8. *The functors $\mathcal{L}oc_{\infty}^\dagger$ and $\Gamma(\mathfrak{X}_\infty, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G_0)_{\theta_0}$ -modules and coadmissible G_0 -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules. The subcategories of coadmissible $D(G)_{\theta_0}$ -modules and coadmissible G -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -modules correspond to each other.*

Proof. We only need to show the second statement. It is clear that a coadmissible $D(G_0)_{\theta_0}$ -module which comes from a coadmissible G -equivariant $\tilde{\mathcal{D}}_{\infty,\mathbb{Q}}^\dagger$ -module is a $D(G)_{\theta_0}$ -module. For the converse, we consider a special vertex $v \in \mathcal{B}$ and a model $\mathfrak{X}(v)$ and the corresponding localisation functor $\mathcal{L}oc_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger$ (where $k = k_{\mathfrak{X}(v)}$) which is an equivalence between finitely presented $D^{an}(\mathbb{G}_v(k)^\circ)_{\theta_0}$ -modules and coherent $\tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger$ -modules on $\mathfrak{X}(v)$. Here, \mathbb{G}_v denotes as before the reductive Bruhat-Tits group scheme over \mathfrak{o} associated with the special vertex v . The adjoint action of G on its Lie algebra induces a ring isomorphism

$$(5.3.9) \quad D^{an}(\mathbb{G}_v(k)^\circ) \xrightarrow{g} D^{an}(\mathbb{G}_{gv}(k)^\circ)$$

for any $g \in G$. Now consider a coadmissible $D(G)_{\theta_0}$ -module M with dual space $V = M'$. We have the family $(\mathcal{M}_{\mathfrak{X}(v)})_{\mathfrak{X}(v)}$ where

$$\mathcal{M}_{\mathfrak{X}(v)} = \mathcal{L}oc_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(M_{k,v}) = \tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger \otimes_{D^{an}(\mathbb{G}_v(k)^\circ)_{\theta_0}} M_{k,v}$$

and $M_{k,v} = (V_{\mathbb{G}_v(k)^\circ-\text{an}})'$ with $k = k_{\mathfrak{X}(v)}$. Let $g \in G$. The map $m \mapsto gm$ on M induces a map $M_{k,v} \rightarrow M_{k,gv}$ which is linear relative to 5.3.9. We therefore have for any open subset $U \subseteq \mathfrak{X}(v)$ a homomorphism

$$\mathcal{M}_{\mathfrak{X}(v)}(U) \xrightarrow{g \cdot} \mathcal{M}_{\mathfrak{X}(gv)}(g \cdot U)$$

which is induced by the map

$$s \otimes m \mapsto (g \cdot s) \otimes gm .$$

for $s \in \tilde{\mathcal{D}}_{\mathfrak{X}(v),k,\mathbb{Q}}^\dagger(U)$, $m \in M_{k,v}$ and where $g \cdot$ is the ring isomorphism 5.3.4. In particular, for an open subset $V \subseteq \mathfrak{X}_\infty$ of the form $V = \text{sp}_{\mathfrak{X}(v)}^{-1}(U)$ with $U \subseteq \mathfrak{X}(v)$ open, this gives a homomorphism for

$$(5.3.10) \quad \mathcal{M}_{\mathfrak{X}(v)}(\text{sp}_{\mathfrak{X}(v)}(V)) \xrightarrow{g \cdot} \mathcal{M}_{\mathfrak{X}(gv)}(g \cdot \text{sp}_{\mathfrak{X}(v)}(V)) = \mathcal{M}_{\mathfrak{X}(gv)}(\text{sp}_{\mathfrak{X}(gv)}(gV))$$

which is linear relative to the ring homomorphism 5.3.5.

A given morphism $\pi : \mathfrak{X}(v') \rightarrow \mathfrak{X}(v)$ with $\mathfrak{X}(v') \in \mathcal{F}(v')$ and $\mathfrak{X}(v) \in \mathcal{F}(v)$ which is an isomorphism on corresponding rigid analytic spaces induces a morphism

$$(5.3.11) \quad \pi_* \mathcal{M}_{\mathfrak{X}(v')} \longrightarrow \mathcal{M}_{\mathfrak{X}(v)}$$

compatible with the morphism of rings 5.3.6 as follows. First of all, one has an isomorphism

$$\pi_* \left(\mathcal{L}oc_{\mathfrak{X}(v'),k',\mathbb{Q}}^\dagger(M_{k',v'}) \right) \xrightarrow{\simeq} \left(\pi_* \tilde{\mathcal{D}}_{\mathfrak{X}(v'),k',\mathbb{Q}}^\dagger \right) \otimes_{D^{an}(\mathbb{G}_{v'}(k')^\circ)_{\theta_0}} M_{k',v'} .$$

Indeed, π_* is exact by 5.2.1 and we may argue with finite presentations as usually. Moreover, we have inclusions $\mathbb{G}_{v'}(k') \subseteq \mathbb{G}_v(k)$ and thus

$$V_{\mathbb{G}_v(k)^\circ-\text{an}} \subseteq V_{\mathbb{G}_{v'}(k')^\circ-\text{an}} .$$

The dual map $M_{k',v'} \rightarrow M_{k,v}$ is linear relative to the natural inclusion

$$D^{an}(\mathbb{G}_{v'}(k')^\circ) \rightarrow D^{an}(\mathbb{G}_v(k)^\circ) .$$

The latter inclusion is compatible with the morphism of rings 5.3.6 via taking global sections. Hence, we have a morphism 5.3.11 as claimed. We now have everything at hand to follow the arguments in the proof of the preceding proposition word for word and to conclude that the projective limit \mathcal{M}_∞ has a G -action which extends its G_0 -action and which makes it a G -equivariant $\tilde{\mathcal{D}}_{\infty, \mathbb{Q}}^\dagger$ -module. This completes the proof of the theorem. \square

We finally look at the special case of the whole adic flag variety \mathcal{X} with its sheaf of infinite order differential operators $\tilde{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^\dagger$, cf. 5.2.11. Recall the functor $\mathcal{L}oc_{\mathcal{X}}^\dagger$ from the end of 5.2.

Theorem 5.3.12. *Let \mathcal{X} be the adic analytic flag variety of \mathbb{G} with its sheaf $\tilde{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^\dagger$ of infinite order differential operators. The functors $\mathcal{L}oc_{\mathcal{X}}^\dagger$ and $\Gamma(\mathcal{X}, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G_0)_{\theta_0}$ -modules and coadmissible G_0 -equivariant $\tilde{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^\dagger$ -modules. The subcategories of coadmissible $D(G)_{\theta_0}$ -modules and coadmissible G -equivariant $\tilde{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^\dagger$ -modules correspond to each other.*

Proof. Taking each $\mathcal{F}(v)$ to be the set of all equivariant formal models, we obtain $\mathfrak{X}_\infty = \mathcal{X}$ according to 5.2.6 and the result follows from the preceding theorem. \square

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